A SUSY GUT of Flavour with $S_4 \times SU(5)$ to NLO

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Abstract

We construct a Supersymmetric (SUSY) Grand Unified Theory (GUT) of Flavour based on $S_4 \times SU(5)$, together with an additional (global or local) Abelian symmetry, and study it to next-to-leading order (NLO) accuracy. The model includes a successful description of quark and lepton masses and mixing angles at leading order (LO) incorporating the Gatto-Sartori-Tonin (GST) relation and the Georgi-Jarlskog (GJ) relations. We study the vacuum alignment arising from F-terms to NLO and such corrections are shown to have a negligible effect on the results for fermion masses and mixings achieved at LO. Tri-bimaximal (TB) mixing in the neutrino sector is predicted very accurately up to NLO corrections of order 0.1%. Including charged lepton mixing corrections implies small deviations from TB mixing described by a precise sum rule, accurately maximal atmospheric mixing and a reactor mixing angle close to three degrees.

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1 Introduction

A long standing quest of theories of particle physics beyond the Standard Model (SM) is to formulate a theory of quark and charged lepton masses and quark mixings. In recent years, this quest has been extended to include the neutrino masses and lepton mixing as a result of tremendous experimental advances and discoveries in neutrino physics. Indeed, perhaps the greatest advance in particle physics over the past dozen years has been the measurement of neutrino masses and mixing involving two large mixing angles associated with atmospheric and solar neutrino oscillation experiments, while the remaining mixing angle, although unmeasured, is constrained by reactor neutrino oscillation experiments to be relatively small. The empirical observation of TB lepton mixing [1] contrasts sharply with the smallness of quark mixing, and this observation, together with the smallness of neutrino masses, provides new and tantalising clues in the search for the origin of quark and lepton flavour in terms of a theory of flavour that would supersede the SM.

TB lepton mixing in particular hints at a spontaneously broken family symmetry G_f which might underpin a flavour theory of all quarks and leptons, but which might only reveal itself in the neutrino sector. What is the nature of such a family symmetry? In the (diagonal) charged lepton mass basis, it has been shown that the neutrino mass matrix leading to TB mixing is invariant under (off-diagonal) transformations S and U which constitute the Klein group [2]. The observed neutrino flavour symmetry corresponding to the two generators S and U of the Klein group may arise either directly or indirectly from certain classes of discrete groups [3]. Several models have been constructed that account for the structure of leptonic mixings, e.g. [4–8], while other models extend the underlying family symmetry to provide a description of the complete fermionic structure [9–27].² If the neutrino flavour symmetry arises directly from the family symmetry [3] then this implies that the family symmetry should contain the generators S and U so that they can be preserved in the neutrino sector at LO. The smallest group that contains the generators S and U together with a (diagonal) phase matrix T is S_4 [2] and the models found in [7, 15, 16, 20, 21] are based on S_4 . The fact that it is possible to construct direct models based on the family symmetry A_4 (generated by S and T only) is owed to the required absence of family symmetry breaking fields (flavons) in the representations 1' and $\mathbf{1}''$ of A_4 . In such A_4 models the symmetry associated with the generator U arises accidentally at LO [6].

Despite the plethora of models, there are surprisingly few which successfully combine a discrete family symmetry containing triplet representations (necessary to account for TB mixing) together with a GUT. Examples are the $A_4 \times SU(5)$ models [17], the $T' \times SU(5)$ model [18], the $A_4 \times SO(10)$ models [19], the $S_4 \times SO(10)$ models [21], the $PSL(2,7) \times SO(10)$ model [22], and the $\Delta_{27} \times SO(10)$ models [25]. The possible combination $S_4 \times SU(5)$ stands out in the sense that it combines the minimal GUT with the minimal choice of family symmetry, which contains the generators S and U.

In this paper we construct a SUSY GUT of Flavour based on $S_4 \times SU(5)$ in which the

¹In a different basis S and U could as well be represented by diagonal matrices.

²See [28] for review papers with more extensive references.

 $\overline{\bf 5}$ matter fields of SU(5) are assigned to a triplet of S_4 , while the 10 matter fields are in a doublet plus a (trivial) singlet of S_4 . The operators are also controlled by an additional U(1) symmetry which segregates different types of flavons into different (charge) sectors at LO, e.g. flavons, whose vacuum expectation values (VEVs) preserve the generators S and U, only couple to neutrinos at LO. Furthermore, the U(1) symmetry controls the amount of flavon contamination between different sectors beyond LO. We shall show that the model predicts TB neutrino mixing very accurately up to corrections of order 0.1% at the GUT scale. In order to do so, we specify the complete effective theory, valid just below the GUT scale, and perform a full operator analysis of all relevant terms including several flavons. Furthermore, we make an exhaustive study of vacuum alignment to NLO arising from the F-terms of driving fields. These fields are, similar to the flavons, gauge singlets which only transform non-trivially under $S_4 \times U(1)$. The model leads to a successful description of quark and charged lepton masses and quark mixing angles, including the GST relation between down and strange quark masses and the Cabibbo angle θ_{12}^q [29], and the GJ relations between charged lepton and down quark masses [30], with bottom-tau Yukawa unification. The GJ factor is also responsible for the (left-handed) charged lepton mixing angle θ_{12}^e being $\theta_{12}^e \approx \theta_{12}^q/3$. Including corrections due to non-zero mixing in the charged lepton sector induces deviations from TB lepton mixing expressed in a lepton mixing sum rule [31] with a reactor mixing angle of order $\theta_{12}^q/(3\sqrt{2})$. Since $\theta_{13}^e\approx 0$ and $\theta_{23}^e \approx 0$, maximal atmospheric mixing holds to good precision at the GUT scale. We note that in the realisation of the model we discuss in detail, small and moderate values of $\tan \beta$, the ratio of the VEVs of the two electroweak Higgs doublets present in the Minimal Supersymmetric Standard Model (MSSM), are preferred because the hierarchy among the top and the bottom quark mass is accounted for by the family symmetry. Since our main concern in this work is the explanation of fermion masses and mixings, we leave aside the problem of constructing a GUT Higgs (super-)potential ensuring the correct breaking of the gauge group SU(5) to the SM.

We remark that an $S_4 \times SU(5)$ model has also been proposed in [20], in which, however, NLO corrections as well as the vacuum alignment of the flavons are not studied in detail. By contrast in the different $S_4 \times SU(5)$ model proposed here the LO predictions are robust against the NLO corrections which are explicitly calculated and shown to be small. Furthermore, the alignment of the flavon VEVs is a natural result of the flavon superpotential. The latter is thoroughly investigated to NLO.

The layout of the remainder of the paper is as follows: in section 2 we define the SUSY $S_4 \times SU(5)$ model for a general class of U(1) charges and discuss the results for fermion masses and mixings at LO. In section 3 we perform an operator analysis of all relevant terms including several flavon fields. In this context, we introduce the notion of desired, dangerous, marginal and irrelevant operators. We find 26 possible U(1) charge assignments which neither lead to dangerous nor to marginal operators. Section 4 contains a study of the vacuum alignment which justifies the alignments assumed in previous sections. On the basis of the results of the analysis of higher-order terms disturbing this alignment and of the possibility to correlate the VEVs of different flavons we choose the actual U(1) charges. In section 5 we discuss the NLO corrections to Yukawa couplings

and to the flavon superpotential for a particular choice of U(1) charges and show that all corrections induced to fermion masses and mixings are small. Section 6 concludes the paper. The first three appendices contain the group theory of S_4 , an example of messengers generating the operators giving rise to the GJ and the GST relations, and the list of dangerous and marginal operators with less than four flavons contributing to the fermion mass matrices, according to the classification introduced in section 3. Appendix D is dedicated to a discussion of how to ensure that the family symmetry is broken in the SUSY limit and how to (further) reduce the number of free parameters among the flavon VEVs introducing additional driving fields and using couplings with positive mass dimension.

2 The $S_4 \times SU(5)$ model and LO results

In this section we present the model and discuss the LO result for fermion masses and mixings. In table 1 we show the superfield charge assignments of our SUSY GUT of Flavour based on $S_4 \times SU(5)$. For convenience the group theory of S_4 is summarised in appendix A. The $\overline{\bf 5}$ matter fields F of SU(5) are assigned to a triplet of S_4 , while the ten-dimensional matter fields are assigned to a doublet T plus the trivial singlet T_3 of S_4 . The right-handed neutrinos N are taken to be a triplet of S_4 , analogous to the A_4 see-saw models in [6], however there are some differences in the neutrino sector, as discussed below. The GUT Higgs fields H_5 , $H_{\overline{5}}$ and $H_{\overline{45}}$ are all singlets under the family symmetry S_4 .³ We note that these Higgs representations each contain a Higgs doublet. The MSSM Higgs doublets H_u and H_d then originate from, respectively, H_5 and one linear combination of the doublets in $H_{\overline{5}}$ and $H_{\overline{45}}$. The $H_{\overline{45}}$ component within H_d is responsible for the GJ relations between charged lepton and down quark masses [30]. Concerning the other (orthogonal) linear combination we assume that it decouples from the low-energy theory by acquiring a GUT scale mass just as the colour triplets, contained in the GUT Higgs fields, and other non-MSSM states [33]. The (light) MSSM Higgs doublets $H_{u,d}$ acquire VEVs $v_{u,d}$ with $\tan \beta = v_u/v_d$.

In addition, we introduce a number of flavon fields Φ_{ρ}^{f} . An important feature of the model is that different flavons couple to different sectors of the theory at LO. The flavons Φ_{ρ}^{f} are labelled both by the representation ρ of S_4 under which they transform $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{3}')$ and by the fermion sector f to which they couple at LO, namely u, d and ν , where $d \sim e$

 $^{^3}$ The SU(5) symmetry might be broken by an additional **24** Higgs field which can be rendered irrelevant for the Yukawa operators by suitable charges under the U(1) symmetry. The **45** Higgs field which should be added due to anomaly cancellation may similarly decouple from the up quark sector. Therefore we disregard these Higgs fields in the following. Since the actual construction of a GUT Higgs (super)potential is beyond the scope of this work, we do not specify further flavon fields which might be necessary in order to allow relevant couplings in this (super-)potential, which were otherwise forbidden by the U(1) symmetry. An alternative possibility to achieve the breaking of the GUT symmetry might arise from appropriately chosen boundary conditions in an extra-dimensional scenario, see [32]. In this case also the problem related to the splitting of doublets and colour triplets is elegantly solved.

⁴ Again, it might be necessary to invoke the presence of further flavons to generate $S_4 \times U(1)$ invariant couplings between $H_{\overline{5}}$ and $H_{\overline{45}}$ in order to introduce mixing among their Higgs doublet components.

Field	T_3	T	F	N	H_5	$H_{\overline{5}}$	$H_{\overline{45}}$	Φ_2^u	$\widetilde{\Phi}_2^u$	Φ_3^d	$\widetilde{\Phi}_3^d$	Φ_2^d	$\Phi^{ u}_{3'}$	$\Phi_2^{ u}$	$\Phi_1^{ u}$
SU(5)	10	10	$\overline{5}$	1	5	$\overline{5}$	$\overline{45}$	1	1	1	1	1	1	1	1
S_4	1	2	3	3	1	1	1	2	2	3	3	2	3′	2	1
U(1)	0	x	y	-y	0	0	z	-2x	0	-y	-x-y-2z	z	2y	2y	2y

Table 1: The symmetries and charges of the superfields in the $SU(5) \times S_4 \times U(1)$ model. The U(1) assignment depends on three integers x, y and z.

up to the difference in the GJ factor. Thus, for example, at LO, the flavon doublet Φ_2^u appears only with TT, the flavon triplet Φ_3^d appears only with FT_3 , while the neutrino flavons Φ_{ρ}^{ν} only appear with NN. Notice that Φ_{ρ}^{ν} consist of singlet, doublet and (primed) triplet representations with vacuum alignments which preserve the generators S and U contained in S_4 leading to TB neutrino mixing.

The segregation of the different flavons, coupling to distinct sectors, at the LO level is achieved through an additional U(1) symmetry. For the time being, we assume this symmetry to be global in order to avoid constraints coming from the requirement of anomaly cancellation. The U(1) charges of the fields are expressed in terms of three integers x, y and z, as shown in table 1. Note that the Higgs fields H_5 and $H_{\overline{5}}$ are taken to be neutral under this symmetry.

The family symmetry S_4 is only broken spontaneously by flavon VEVs in our model. On the other hand, the spontaneous breakdown of the global U(1) symmetry leads to the appearance of a (very light) Goldstone boson unless the U(1) is also explicitly broken. For this reason we assume a scenario in which the U(1) symmetry is explicitly broken in the hidden sector of the theory which is also responsible for SUSY breaking, so that the soft terms do not respect the U(1) symmetry. Then the would-be Goldstone boson will have a mass of the order of the soft SUSY mass scale of around 1 TeV.⁵ Alternatively one could gauge the U(1) symmetry and add further particles in order to cancel the anomalies. As the set of these additional particles would depend on the explicit U(1) charge assignments, we do not follow this approach.

In our study, we disregard possible corrections to fermion masses and mixings which are due to deviations from the canonical normalisation of the Kähler potential. Such deviations arise in general, if subleading corrections involving (several) flavons are taken into account. Studies of the possible effects of non-canonically normalised kinetic terms on fermion masses and mixings can be found in, e.g., [34].

The lowest dimensional Yukawa operators invariant under the family symmetry $S_4 \times$

 $^{^5}$ We remark that in this context it would be interesting to investigate whether the family symmetry S_4 alone is sufficient to constrain the soft mass terms of sfermions in such a way that all bounds associated with flavour changing neutral current and lepton flavour violating processes can be satisfied without tuning the soft mass parameters.

U(1), contributing to the up quark mass matrix, are (we omit order one coefficients in the following)

$$T_3 T_3 H_5 + \frac{1}{M} T T \Phi_2^u H_5 + \frac{1}{M^2} T T \Phi_2^u \widetilde{\Phi}_2^u H_5 . \tag{2.1}$$

By M we denote a generic messenger scale which is common for all higher-dimensional operators we discuss. It is expected to be around the GUT scale. If not explicitly stated, we take into account all possible independent S_4 contractions for each operator so that frequently one given operator entails - depending on the vacuum alignment of the flavons - a certain number of different contributions to the fermion mass matrices.

The LO operators giving rise to masses for charged leptons and down quarks read

$$\frac{1}{M}FT_3\Phi_3^d H_{\overline{5}} + \frac{1}{M^2}(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1 H_{\overline{45}} + \frac{1}{M^3}(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3 H_{\overline{5}} , \qquad (2.2)$$

where $(\cdots)_1$ and $(\cdots)_3$ denote the contraction to an S_4 invariant 1 and to the triplet 3, respectively. Note that there are other possible operator contractions involving the same fields that we do not write down. As a first step towards achieving the GJ and GST relations we have assumed that the two contractions shown in Eq. (2.2) are the dominant ones among the various possible ones, existing in a generic effective theory with a cutoff scale M. One example of messengers which only give rise to these contractions is discussed in detail in appendix B and shown diagrammatically in figure 1. The operator involving the Higgs field $H_{\overline{45}}$ must have the appropriate form,

$$\frac{1}{M^2} \left(F_1 \widetilde{\Phi}_{3,1}^d + F_2 \widetilde{\Phi}_{3,3}^d + F_3 \widetilde{\Phi}_{3,2}^d \right) \left(T_1 \Phi_{2,2}^d + T_2 \Phi_{2,1}^d \right) H_{\overline{45}} , \qquad (2.3)$$

to give rise to the GJ relations, $m_d = 3m_e$ and $m_s = m_{\mu}/3$ and $m_b = m_{\tau}$, after insertion of the flavon VEVs. The third operator in Eq. (2.2) leads to

$$\frac{x_1}{M^3} \Phi_{2,1}^d \Phi_{2,2}^d [F_1(T_1 \widetilde{\Phi}_{3,2}^d + T_2 \widetilde{\Phi}_{3,3}^d) + F_2(T_1 \widetilde{\Phi}_{3,1}^d + T_2 \widetilde{\Phi}_{3,2}^d) + F_3(T_1 \widetilde{\Phi}_{3,3}^d + T_2 \widetilde{\Phi}_{3,1}^d)] H_{\overline{5}}
+ \frac{x_2}{M^3} [(F_2(\Phi_{2,2}^d)^2 + F_3(\Phi_{2,1}^d)^2)(T_1 \widetilde{\Phi}_{3,2}^d + T_2 \widetilde{\Phi}_{3,3}^d) + (F_3(\Phi_{2,2}^d)^2 + F_1(\Phi_{2,1}^d)^2)(T_1 \widetilde{\Phi}_{3,1}^d + T_2 \widetilde{\Phi}_{3,2}^d)
+ (F_1(\Phi_{2,2}^d)^2 + F_2(\Phi_{2,1}^d)^2)(T_1 \widetilde{\Phi}_{3,3}^d + T_2 \widetilde{\Phi}_{3,1}^d)] H_{\overline{5}} ,$$
(2.4)

where the two coupling constants x_1 and x_2 indicate two independent invariants. With an appropriate vacuum alignment of the flavon VEVs, Eq. (2.4) gives rise to equal (12) and (21) entries in the mass matrices necessary to achieve the GST relation.

In the neutrino sector the leading terms are

$$y_D F N H_5 + \alpha N N \Phi_1^{\nu} + \beta N N \Phi_2^{\nu} + \gamma N N \Phi_{3'}^{\nu} . \tag{2.5}$$

Using the vacuum alignment

$$\langle \Phi_2^u \rangle = \varphi_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \langle \widetilde{\Phi}_2^u \rangle = \widetilde{\varphi}_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$
 (2.6)

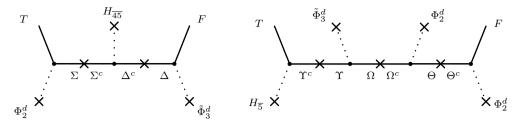


Figure 1: The Feynman diagram on the left shows how the contribution $(T\Phi_2^d)_1(F\widetilde{\Phi}_3^d)_1H_{\overline{45}}/M^2$ can arise in the context of a high energy completion. On the right we show the relevant diagram for generating the effective operator $(T\widetilde{\Phi}_3^d)_3(F\Phi_2^d\Phi_2^d)_3H_{\overline{5}}/M^3$. In these diagrams scalars/fermions are displayed by dotted/solid lines. Crosses indicate a VEV for scalar components and mass insertions for fermions. See appendix B for details.

for the two flavons in doublet representations coupling to up quarks we see that the up quark mass matrix M_u is diagonal

$$M_u \approx \begin{pmatrix} \varphi_2^u \widetilde{\varphi}_2^u / M^2 & 0 & 0 \\ 0 & \varphi_2^u / M & 0 \\ 0 & 0 & 1 \end{pmatrix} v_u .$$
 (2.7)

Taking

$$\varphi_2^u/M \approx \lambda^4 \text{ and } \widetilde{\varphi}_2^u/M \approx \lambda^4,$$
 (2.8)

with $\lambda \approx 0.22$ being the Wolfenstein parameter [35], we obtain the well-known mass hierarchy among the up quarks

$$m_u: m_c: m_t \approx \lambda^8: \lambda^4: 1. \tag{2.9}$$

Similarly, we see using the alignment

$$\langle \Phi_3^d \rangle = \varphi_3^d \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \langle \widetilde{\Phi}_3^d \rangle = \widetilde{\varphi}_3^d \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} , \qquad \langle \Phi_2^d \rangle = \varphi_2^d \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad (2.10)$$

that the down quark, M_d , and charged lepton mass matrix, M_e , are at LO of the form (in the convention in which left-handed fields are on the left-hand side and right-handed fields on the right-hand side of the mass matrix)

$$M_{d} \approx \begin{pmatrix} 0 & (\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & -(\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} \\ -(\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & \varphi_{2}^{d} \widetilde{\varphi}_{3}^{d} / M^{2} & -\varphi_{2}^{d} \widetilde{\varphi}_{3}^{d} / M^{2} + (\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} \\ 0 & 0 & \varphi_{3}^{d} / M \end{pmatrix} v_{d} , \qquad (2.11)$$

and

$$M_{e} \approx \begin{pmatrix} 0 & -(\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & 0 \\ (\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & -3\varphi_{2}^{d} \widetilde{\varphi}_{3}^{d} / M^{2} & 0 \\ -(\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & 3\varphi_{2}^{d} \widetilde{\varphi}_{3}^{d} / M^{2} + (\varphi_{2}^{d})^{2} \widetilde{\varphi}_{3}^{d} / M^{3} & \varphi_{3}^{d} / M \end{pmatrix} v_{d} . \tag{2.12}$$

 v_d denotes the VEV of the electroweak Higgs field H_d which is in general a linear combination of the doublet components of the GUT Higgs fields $H_{\overline{5}}$ and $H_{\overline{45}}$. Assuming that the angle associated with this mixing is of order one we can absorb the corresponding order one factors into the other (not displayed) order one coefficients of each operator. For

$$\varphi_3^d/M \approx \lambda^{1+k} , \qquad \widetilde{\varphi}_3^d/M \approx \lambda^{2+k} , \qquad \varphi_2^d/M \approx \lambda , \qquad (2.13)$$

with k=0 or k=1, we find for the down quark and charged lepton mass hierarchy

$$m_d: m_s: m_b \approx \lambda^4: \lambda^2: 1 , \qquad (2.14)$$

$$m_e: m_\mu: m_\tau \approx (1/3) \,\lambda^4: 3 \,\lambda^2: 1 \,,$$
 (2.15)

and for the mixing angles θ^d_{ij} and θ^e_{ij} of the left-handed fields

$$\theta_{12}^d \approx \lambda \ , \ \theta_{13}^d \approx \lambda^3 \ , \ \theta_{23}^d \approx \lambda^2 \ ,$$
 (2.16)

$$\theta_{12}^e \approx (1/3) \lambda \ , \ \theta_{13}^e \approx 0 \ , \ \theta_{23}^e \approx 0 \ .$$
 (2.17)

The mass of the third generation of charged leptons and down quarks is at LO given by

$$m_b \approx m_\tau \approx \lambda^{1+k} v_d$$
 (2.18)

As a consequence, the two possible choices of k are equivalent to two different types of models: for k=0 we have $m_b \approx m_\tau \approx 40/\tan\beta \,\mathrm{GeV}$ so that the value of $\tan\beta$ is expected to be larger than 30, while for k=1 smaller values of tan β in the range $5 \lesssim \tan \beta \lesssim 15$ are preferred. Since the up quark mass matrix is diagonal, see Eq. (2.7), the Cabibbo angle has to be generated in the down quark sector, as one can see from Eq. (2.16). Also the two other quark mixing angles $\theta_{13,23}^q \approx \theta_{13,23}^d$ turn out to be of the correct order of magnitude. Furthermore, the model incorporates the GST relation $\theta_{12}^q \approx \theta_{12}^d \approx \sqrt{m_d/m_s}$, see Eqs. (2.14,2.16), arising from the equality of the (12) and (21) elements as well as the vanishing of the (11) element in the down quark mass matrix M_d at LO. Note that we can achieve the same LO results in the down quark and charged lepton sector if we assume $\langle \widetilde{\Phi}_3^d \rangle = \widetilde{\varphi}_3^d (0, \kappa, 1)^t$ with $|\kappa| = 1$, κ complex, instead of using $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$ as shown in Eq. (2.10). For this reason we perform the study of marginal and dangerous Yukawa operators, which can be found in the next section, assuming the alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, \kappa, 1)^t$. However, as one can see the flavon superpotential, discussed in section 4, only gives rise to the alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$. Thus, in the actual realisation, given in section 5, the latter alignment is used.

Finally, we display the LO results for the neutrino sector: the neutrino Dirac mass matrix M_D arising from Eq. (2.5) has a very simple form

$$M_D = y_D \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u , \qquad (2.19)$$

while the right-handed neutrino mass matrix

$$M_{R} = \begin{pmatrix} \alpha \varphi_{1}^{\nu} + 2 \gamma \varphi_{3'}^{\nu} & \beta \varphi_{2}^{\nu} - \gamma \varphi_{3'}^{\nu} & \beta \varphi_{2}^{\nu} - \gamma \varphi_{3'}^{\nu} \\ \beta \varphi_{2}^{\nu} - \gamma \varphi_{3'}^{\nu} & \beta \varphi_{2}^{\nu} + 2 \gamma \varphi_{3'}^{\nu} & \alpha \varphi_{1}^{\nu} - \gamma \varphi_{3'}^{\nu} \\ \beta \varphi_{2}^{\nu} - \gamma \varphi_{3'}^{\nu} & \alpha \varphi_{1}^{\nu} - \gamma \varphi_{3'}^{\nu} & \beta \varphi_{2}^{\nu} + 2 \gamma \varphi_{3'}^{\nu} \end{pmatrix},$$
 (2.20)

is the origin of TB mixing in this model if we use the vacuum alignment

$$\langle \Phi_{3'}^{\nu} \rangle = \varphi_{3'}^{\nu} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \langle \Phi_{2}^{\nu} \rangle = \varphi_{2}^{\nu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \langle \Phi_{1}^{\nu} \rangle = \varphi_{1}^{\nu}$$
 (2.21)

(together with the information that these VEVs are all of the same order of magnitude). Applying the type I see-saw formula

$$m_{\nu}^{eff} = M_D M_R^{-1} M_D^t , \qquad (2.22)$$

we find for the (complex) light neutrino masses

$$m_1 = \frac{y_D^2 v_u^2}{\alpha \varphi_1^{\nu} - \beta \varphi_2^{\nu} + 3\gamma \varphi_{3'}^{\nu}} , \quad m_2 = \frac{y_D^2 v_u^2}{\alpha \varphi_1^{\nu} + 2\beta \varphi_2^{\nu}} , \quad m_3 = \frac{y_D^2 v_u^2}{-\alpha \varphi_1^{\nu} + \beta \varphi_2^{\nu} + 3\gamma \varphi_{3'}^{\nu}} . \quad (2.23)$$

Due to the three different couplings α , β , γ the three light neutrino masses are unrelated and any type of mass hierarchy can be accommodated. Especially, the former are not constrained by a sum rule, as it is the case in the A_4 models [6].

Concerning the approximate scale of the VEVs of the flavons Φ_1^{ν} , Φ_2^{ν} and $\Phi_{3'}^{\nu}$ we note that, since they set the scale for right-handed neutrino masses, the physical neutrino masses

$$m_i \sim 0.1 \text{ eV} \sim \frac{y_D^2 v_u^2}{\varphi_{1/2/3'}^2} ,$$
 (2.24)

imply that, $\varphi_{1,2,3'}^{\nu} \sim 10^{13}$ GeV, assuming $y_D \sim 0.3$ and $\tan \beta \sim 10$ for example. Assuming the generic messenger scale M to be of the order of the GUT scale, $M \approx 10^{16}$ GeV, we see that $\varphi_{1,2,3'}^{\nu}$ fulfil

$$\varphi_1^{\nu} \approx \lambda^4 M , \qquad \varphi_2^{\nu} \approx \lambda^4 M , \qquad \varphi_{3'}^{\nu} \approx \lambda^4 M .$$
 (2.25)

The neutrino mixing stemming from Eq. (2.22) is exactly TB mixing. However, it will be corrected by the non-trivial (12) mixing present in the charged lepton sector, see Eq. (2.17), so that the lepton mixing angles at the high energy scale are given by [31],

$$\sin^2 \theta_{23}^l \approx 1/2 \; , \; \sin^2 \theta_{12}^l \approx 1/3 + 2/9 \, \lambda \cos \delta^l \; , \; \sin \theta_{13}^l \approx \lambda/(3\sqrt{2}) \; ,$$
 (2.26)

which incorporates the usual prediction for the reactor mixing angle, associated with the presence of the GJ factor and the SU(5) context, leading to the prediction $\theta_{13}^l \approx 3^\circ$ for $\lambda \approx 0.22$, and, after eliminating λ , to the sum rule relation [31],

$$\sin^2 \theta_{12}^l \approx \frac{1}{3} \left(1 + 2\sqrt{2} \sin \theta_{13}^l \cos \delta^l \right), \tag{2.27}$$

where δ^l is the leptonic Dirac CP phase.

The neutrino sector in the S_4 model above differs from that in the A_4 one [6] by the presence of the doublet flavon Φ_2^{ν} whose VEV structure preserves the generators S and U. Note that, if the (irreducible) representations of S_4 are decomposed into those of its subgroup A_4 , we find that the doublet Φ_2^{ν} decomposes into the two non-trivial singlets $\mathbf{1}'$ and $\mathbf{1}''$, see appendix A. In the A_4 model separate flavons in representations $\mathbf{1}'$ and $\mathbf{1}''$, respectively, which, if allowed to appear with independent couplings, would violate the symmetry associated with the generator U, have to be absent in order to achieve TB mixing [6]. This is why the A_4 model accidentally preserves the generator U in the neutrino sector, even though U is not contained in the group A_4 [6]. In the present model, both the generators S and U are contained in S_4 , and remain preserved in the neutrino sector at LO, so that the neutrino flavour symmetry is reproduced in a more direct way.

The vacuum alignment of the flavons which has been only assumed in this section will be discussed in more detail in section 4. We will show that the alignment (in which $\langle \tilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$) can be produced through F-terms of an additional set of gauge singlet fields charged under the family symmetry $S_4 \times U(1)$. Regarding the assumed sizes of the VEVs we find that these can be partly explained by the superpotential which gives rise to correlations among the VEVs and partly by introducing additional gauge singlets which allow couplings of positive mass dimension in the flavon superpotential, whose magnitude can be appropriately chosen in order to reproduce the sizes of the VEVs. This issue is discussed in section 5.1 and appendix D.

3 Dangerous and marginal Yukawa operators

After presenting the LO result which incorporates the prediction of TB mixing in the neutrino sector and the successful accommodation of all charged fermion masses and quark mixings, we now discuss in more detail the role of the additional U(1) symmetry in forbidding all operators which would otherwise have a considerable effect on these LO results. For example, as already remarked, beyond the LO we expect the segregation of different flavons Φ^f associated with a particular quark and lepton type $f = u, d, \nu$ to break down.

In order to identify operators which should be forbidden, we first classify them according to which contributions they give to the fermion mass matrices. In this analysis we assume for the VEVs of the flavons to have the LO form, as shown in Eqs. (2.6,2.10,2.21) together with the generalised alignment of $\langle \widetilde{\Phi}_3^d \rangle$. However, as we will discuss below, these VEVs receive in general corrections stemming from subleading terms present in the flavon superpotential. We fix the actual values of the U(1) charges x, y and z on the basis of the results for the flavon superpotential. For this particular choice (and the specific alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$) we discuss, in section 5, the subleading corrections induced through shifts in the flavon VEVs (and subleading operators), showing that their effects on fermion masses and mixings are negligible.

We can distinguish the following four types of Yukawa operators

- desired operators: These are the operators which by definition of the U(1) charges are present at LO, see Eqs. (2.1,2.2,2.5).
- dangerous operators: These operators strongly perturb the form of the mass matrices achieved at LO. In the case of charged fermions their contribution is larger than the one stemming from the desired operators. In the case of right-handed neutrinos, any contribution which is larger than or of the same order of magnitude in λ as the one coming from the desired operators has to be considered as dangerous because TB mixing crucially depends on the form of the right-handed neutrino mass matrix as well as on the fact that all entries of the latter are of the same order of magnitude in λ .
- marginal operators: These operators give contributions to charged fermion mass matrices which are of the same order in λ as the LO contribution. Although not dangerous in the above sense, their presence has a significant impact on the final result. For example, in the case of the GST relation it might happen that such a marginal operator contributes differently to the (12) and the (21) elements of the down quark mass matrix M_d so that the relation between the Cabibbo angle and the masses of down and strange quark is lost.
- irrelevant operators: These operators do not contribute to fermion masses or mixings at LO in λ and thus do not need to be forbidden. For phenomenology they are however not completely negligible, since they (can) give rise to corrections to the LO result, e.g. they are responsible for deviations from exact TB mixing in the neutrino sector.

According to this classification we wish to forbid all dangerous and all marginal operators. Since the entries of the mass matrices M_u , $M_{d,e}$, M_D and M_R are of different order in λ , we list the structures of the LO as well as the dangerous and the marginal contributions for each sector separately. Note that we constrain ourselves in this study to the case k=1, since it turns out that this choice reduces the number of dangerous and marginal operators to a certain extent and thus facilitates the search for appropriate U(1) charge combinations x, y and z, especially with small absolute values. The value of k is thus specified to k=1 for the rest of the paper.

In the up quark sector, Eq. (2.7) tells us that the LO mass matrix has the form

$$M_u^{\rm LO} \sim \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$
 (3.1)

so that we classify as dangerous (dang) all mass matrix entries which are equal or larger than

$$M_u^{\text{dang}} \gtrsim \begin{pmatrix} \lambda^7 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^3 & \lambda \\ \lambda^3 & \lambda & . \end{pmatrix}$$
 (3.2)

Since the (33) entry of M_u^{LO} is $\mathcal{O}(1)$ any corrections to this entry are irrelevant. The sizes of the other diagonal entries are determined by the requirement of not having too large up and charm quark masses, while the bounds on the off-diagonal elements originate from the constraints on the quark mixing angles, $\theta_{12}^q \approx \lambda$, $\theta_{23}^q \approx \lambda^2$ and $\theta_{13}^q \approx \lambda^3$, as well as from achieving the correct mass hierarchy. Similarly, the operators characterised as marginal (marg) give rise to entries in M_u of the order

$$M_u^{\text{marg}} \sim \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & . \end{pmatrix}$$
 (3.3)

Using Eqs. (2.11,2.12) we see that the LO of the entries in $M_{d,e}$ is

$$M_{d,e}^{\rm LO} \sim \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ \lambda^5 & \lambda^4 & \lambda^2 \end{pmatrix} . \tag{3.4}$$

Thus, we classify operators as dangerous and marginal which lead to the following mass matrix entries

$$M_{d,e}^{\mathrm{dang}} \gtrsim \begin{pmatrix} \lambda^5 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda \end{pmatrix} \quad \text{and} \quad M_{d,e}^{\mathrm{marg}} \sim \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ \lambda^5 & \lambda^4 & \lambda^2 \end{pmatrix} ,$$
 (3.5)

respectively. Note that our results are based on the assumption that the mass matrices are symmetric regarding the order of magnitude in λ . We make this assumption although the off-diagonal elements in the third row and column of M_d and M_e , at LO, are non-zero only for one of the two matrices but not for both simultaneously. Note further that the (11) entry of $M_{d,e}$ vanishes at LO. The constraint on this entry to be smaller than λ^5 results from the requirement that the determinant of $M_{d,e}$ should not exceed λ^{12} .

In the neutrino sector, all operators involving flavons, which contribute to the neutrino Dirac mass matrix M_D ,

$$M_D^{\rm LO} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$
 (3.6)

can be classified as irrelevant, because the LO term, see Eq. (2.5), originates at the renormalisable level, i.e. does not require the presence of any flavons. As already explained, since the form of the LO result of M_R is crucial to achieve TB neutrino mixing,

$$M_R^{\rm LO} \sim \begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^4 & \lambda^4 \end{pmatrix} , \qquad (3.7)$$

any further contribution being of order λ^4 or larger is associated with a dangerous operator

$$M_R^{\rm dang} \gtrsim \begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^4 & \lambda^4 \end{pmatrix} .$$
 (3.8)

All other operators contributing at the level $\lesssim \lambda^5$ are irrelevant.

Any operator comprising two superfields of the type T_3 , T, F and N and an arbitrary number of flavon fields that gives a dangerous or marginal contribution to a mass matrix should be forbidden by the additional U(1) symmetry. In the following, we first classify all operators with up to three flavons according to the categories above, because the LO result for fermion masses and mixings is generated by operators with at maximum three flavons, see Eq. (2.2). The structures of the resulting mass matrices determine the unwanted operators which are listed in appendix C. Note that in this calculation we assumed the vacuum alignment of the flavons as given in Eqs. (2.6,2.10,2.21), apart from the fact that we allow $\langle \widetilde{\Phi}_3^d \rangle$ to be aligned as $(0, \kappa, 1)^t$ with $|\kappa| = 1$, κ complex, instead of using $(0, -1, 1)^t$ as shown in Eq. (2.10). The reason for this slightly generalised alignment of $\langle \widetilde{\Phi}_3^d \rangle$ lies in the fact that keeping the relative phase among the two nonvanishing entries of $\langle \widetilde{\Phi}_3^d \rangle$ arbitrary might leave us more freedom in the construction of the flavon superpotential, from which the alignment of the flavons originates. Again, we emphasise that using the actual realisation of the flavon (super-)potential presented in section 4, we arrive at the alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$. However, an analysis of the Yukawa operators in the slightly more general framework is still useful, because in any case the solutions found in this analysis can also be applied to the specific alignment of $\langle \Phi_3^d \rangle$ in which κ is fixed to a certain value. As we comment below, fixing κ to -1 leads to some more possible sets of charges x, y and z, which however do not give rise to any feature not already revealed in the sets found through the analysis using the generalised alignment of $\langle \Phi_3^d \rangle$. Apart from the unwanted operators the table in appendix C also shows the corresponding λ -suppression as well as the entries of the mass matrices which are in conflict with the LO setup. 6 Entries for which the operator is marginal in the above sense are marked with square brackets, whereas in all other cases the operator is dangerous. The three operators denoted with a prime (43', 48', 54') differ from the LO terms of the down quark sector in Eq. (2.2) only by the exchange of $H_{\overline{5}}$ and $H_{\overline{45}}$. All other terms given for the down quark sector must be forbidden for both Higgs fields, $H_{\overline{5}}$ as well as $H_{\overline{45}}$.

A complete scan over the parameters x, y, z with $|x|, |y|, |z| \leq 5$ yields 43 different U(1) symmetries which forbid all unwanted operators with up to three flavon fields. Here we have identified the U(1) symmetry related to (-x, -y, -z) with the one represented by (x, y, z). Apart from this also dangerous or marginal operators with more than three flavons should be forbidden. The dangerous operators are

$$TTH_5(\Phi_2^d)^4/M^4 \ , \qquad TTH_5(\Phi_3^d)^3\Phi_2^d/M^4 \ , \qquad TTH_5(\Phi_2^d)^3\Phi_2^\nu/M^4 \ ,$$

 $TTH_5(\Phi_2^d)^3(\Phi_3^d)^2/M^5$, $TTH_5(\Phi_2^d)^7/M^7$, $FTH_{\overline{5},\overline{45}}\Phi_3^d(\Phi_2^d)^3/M^4$, $NN(\Phi_2^d)^4/M^3$.

As marginal operators we find

$$TTH_5(\Phi_3^d)^2(\Phi_2^d)^2/M^4$$
, $TTH_5(\Phi_2^d)^2(\widetilde{\Phi}_3^d)^2/M^4$, $TTH_5(\Phi_3^d)^2\widetilde{\Phi}_3^d\Phi_2^d/M^4$,

⁶Note that we only give one of the two entries (ij) and (ji) in the case of the symmetric or symmetrised terms TTH_5 , T_3TH_5 , NN.

#	1	2	3	4	5	6	7	8	9	10	11	12	13
x	1	2	2	2	2	2	3	4	4	4	4	5	5
y	2	1	1	4	5	5	1	2	3	5	5	1	4
z	5	4	5	5	1	4	5	5	5	2	3	4	1
#	14	15	16	17	18	19	20	21	22	23	24	25	26
x	-1	1	2	2	2	2	-3	3	3	4	4	4	5
y	3	-4	-5	-3	4	5	4	-2	-2	-5	-1	-1	-2
z	5	5	4	5	-5	-3	5	4	5	3	2	5	4

Table 2: The 26 viable U(1) symmetries defined by the parameters x,y,z ($|x|,|y|,|z| \leq 5$) for the alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0,\kappa,1)^t$ with $|\kappa|=1, \kappa$ complex. Obviously, for each set of charges (x,y,z) also the set (-x,-y,-z) is a viable candidate.

$$TTH_5\Phi_3^d(\Phi_2^d)^2\Phi_{3'}^{\nu}/M^4 \ , \qquad TTH_5\Phi_3^d\widetilde{\Phi}_3^d(\Phi_2^d)^3/M^5 \ , \qquad TTH_5(\Phi_2^d)^4\Phi_1^{\nu}/M^5 \ , \\ TTH_5(\Phi_2^d)^6/M^6 \ , \qquad FTH_{\overline{5}}_{\overline{45}}\widetilde{\Phi}_3^d(\Phi_2^d)^3/M^4 \ .$$

Obviously, these must be removed as well,⁷ so that we end up with 26 viable U(1) symmetries listed in table 2. This set will serve as a source of a candidate U(1) symmetry which eventually leads to a successful $S_4 \times SU(5)$ model.

Assuming the alignment of $\langle \widetilde{\Phi}_3^d \rangle$ to be the one as given in Eq. (2.10), we find that two operators among those classified as dangerous or marginal become irrelevant, namely operators #18 and #32 in the table found in appendix C. Allowing these two operators, we find 18 additional solutions for the U(1) charges x, y and z as compared to the 43 mentioned above. Including eventually the requirement to forbid the dangerous and marginal operators with more than three flavons leaves us with 15 new sets (x, y, z) that are added to the 26 U(1) symmetries of table 2. However, as we do not find any set with charges x, y and z with |x|, |y|, |z| < 4, these 15 new solutions are qualitatively not different from the ones given in table 2, so that we do not consider them any further. Nevertheless in the subsequent sections 4 and 5 the alignment of $\langle \widetilde{\Phi}_3^d \rangle$ is fixed through the flavon superpotential to be proportional to $(0, -1, 1)^t$.

Finally, we remark that the high energy completion we proposed in order to only generate the operators $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$ and $(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3H_{\overline{5}}/M^3$ in the down quark sector actually depends on the choice of the combination x,y and z, because in the calculation for generic charges x,y and z we implicitly relied on the fact that all heavy fields appearing as messengers carry (different) charges under the U(1) symmetry so that only the operators given in appendix B are generated at the renormalisable level. This must be taken into account as an additional constraint on the solutions presented in this section. We will comment on this point in section 5 and appendix B.

⁷We remark that the classification of these operators into dangerous and marginal does not depend on the relative phase introduced in the generalised alignment of $\langle \Phi_3^d \rangle$.

4 Vacuum alignment

The origin of the vacuum alignment is an integral part of a model of fermion masses and mixings using a non-Abelian family symmetry. We first discuss in section 4.1 how to achieve the vacuum alignment shown in Eqs. (2.6,2.10,2.21) by introducing a new set of fields, called driving fields in the following, from whose F-terms we derive the alignment. We actually show that in this case $\langle \Phi_3^d \rangle \propto (0, -1, 1)^t$ is the only solution, so that the parameter κ in the generalised form of the alignment of $\langle \widetilde{\Phi}_3^d \rangle$, used in the preceding section, is fixed to $\kappa = -1$. The U(1) charges of the driving fields are given in terms of the three parameters x, y and z which have been introduced in section 2. The additionally allowed operators of the flavon superpotential beyond those given in section 4.1 are then determined for all 26 sets of U(1) charges x, y and z shown in table 2. On the basis of this study we exclude all sets (x, y, z) for which these additional operators strongly perturb the LO vacuum alignment. Focusing on the remaining four choices of U(1) charges x, yand z for which no such operators arise if the LO results of the flavon VEVs are used, we search for possibilities to (partly) correlate the flavon VEVs by introducing further driving fields. We eventually fully specify the values of the U(1) charges x, y and z by choosing the possibility which allows for the largest number of correlations among the scales of the various flavon VEVs. This is explained in section 4.3 and in detail shown in appendix D. Furthermore, we discuss in section 5.1 and appendix D that a minimum of undetermined parameters among the flavon VEVs can be reached, if driving fields are included which allow for couplings with positive mass dimension.

4.1 Flavon superpotential at LO

In our approach we generate the vacuum alignment through F-terms by coupling the flavons to driving fields. The latter are - similar to the flavons - gauge singlets and transform in general in a non-trivial way under $S_4 \times U(1)$. We introduce furthermore a $U(1)_R$ symmetry under which all driving fields carry charge +2. In contrast to this, flavons and the GUT Higgs fields are uncharged under $U(1)_R$ and supermultiplets containing SM fields (or right-handed neutrinos) have $U(1)_R$ charge +1. In this way, the driving fields can only appear linearly in the superpotential and in addition do not have direct interactions with SM fermions (and right-handed neutrinos). Under the assumption that the family symmetry $S_4 \times U(1)$ is broken at high energies, a scale at which SUSY is not broken in the visible sector, we can deduce the alignment of the flavon VEVs from the equations arising from setting the F-terms of the driving fields to zero. Table 3 gives a list of driving fields with which we generate the vacuum alignment in Eqs. (2.6,2.10,2.21). The U(1)charges are expressed in terms of the parameters x, y and z so as to allow the relevant superpotential operators which give rise to the desired alignments. In the following we will discuss these terms in turn. Most of the alignments are achieved through renormalisable operators with three fields in order not to introduce further mass scales. In the case of

⁸There are also other possibilities to break a family symmetry, e.g. through non-trivial boundary conditions in extra-dimensional models [36].

Driving field	X_1^d	Y_2^d	$Z^{ u}_{3'}$	Y_2^{ν}	\overline{X}_1^d	$X^{\nu d}_{1'}$	Y_2^{du}	X_1^u
SU(5)	1	1	1	1	1	1	1	1
S_4	1	2	3′	2	1	1′	2	1
U(1)	-2z	2y - 2z	-4y	-4y	x + 2y + z	x-y+2z	2x-z	2x

Table 3: The driving fields required for obtaining the vacuum alignment. All these fields carry charge +2 under $U(1)_R$.

non-renormalisable terms we suppress the operators by appropriate powers of the generic messenger scale M. Note that in such a setup with no superpotential couplings of positive mass dimension it is impossible to exclude the trivial solution, i.e. a vacuum in which all flavon VEVs vanish. However, having chosen the specific set of U(1) charges x, y and z we comment on this issue in section 5.1 and present a way to enforce spontaneous family symmetry breaking in appendix D.

The driving field X_1^d , coupled to Φ_2^d through

$$X_1^d(\Phi_2^d)^2 = X_1^d\Phi_{21}^d\Phi_{22}^d \,, \tag{4.1}$$

allows to align $\langle \Phi_2^d \rangle$ either as

$$\langle \Phi_2^d \rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or as } \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$
 (4.2)

In the following we choose the alignment in which the component $\Phi_{2,1}^d$ receives a non-zero VEV. Assuming Eq. (2.13) to hold, the alignment of the VEV of Φ_2^d is generated through an operator of the order λ^2 .

Using the field Y_2^d and the alignment achieved for $\langle \Phi_2^d \rangle$ we align the VEV of Φ_3^d . In general we find three independent dimension-5 terms coming from $Y_2^d(\Phi_2^d)^2(\Phi_3^d)^2/M^2$

$$\frac{1}{M^{2}}((\Phi_{3,1}^{d})^{2} + 2\Phi_{3,2}^{d}\Phi_{3,3}^{d})(Y_{2,1}^{d}(\Phi_{2,1}^{d})^{2} + Y_{2,2}^{d}(\Phi_{2,2}^{d})^{2})$$

$$+ \frac{1}{M^{2}}\Phi_{2,1}^{d}\Phi_{2,2}^{d}\left[Y_{2,1}^{d}((\Phi_{3,3}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,2}^{d}) + Y_{2,2}^{d}((\Phi_{3,2}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,3}^{d})\right]$$

$$+ \frac{1}{M^{2}}\left[Y_{2,1}^{d}(\Phi_{2,2}^{d})^{2}((\Phi_{3,2}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,3}^{d}) + Y_{2,2}^{d}(\Phi_{2,1}^{d})^{2}((\Phi_{3,3}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,2}^{d})\right] ,$$
(4.3)

which yield the following conditions

$$(\Phi_{3,1}^d)^2 + 2\Phi_{3,2}^d \Phi_{3,3}^d = 0$$
 and $(\Phi_{3,3}^d)^2 + 2\Phi_{3,1}^d \Phi_{3,2}^d = 0$, (4.4)

if the alignment of $\langle \Phi_2^d \rangle$ is plugged into the *F*-terms of $Y_{2,1}^d$ and $Y_{2,2}^d$. Eq. (4.4) shows that $\langle \Phi_3^d \rangle$ has to be aligned as

$$\langle \Phi_3^d \rangle \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or as } \frac{1}{3} \begin{pmatrix} 2\omega^p \\ -1 \\ 2\omega^{-p} \end{pmatrix}, \ p = 0, \pm 1 \ ,$$
 (4.5)

with $\omega = e^{2\pi i/3}$. As before, we select the first of these four possible alignments. Assuming the relative size of the VEVs φ_2^d and φ_3^d with respect to the messenger scale M as given in Eq. (2.13) we find that the operators responsible for the alignment of $\langle \Phi_3^d \rangle$ arise at the level λ^6 .

The vacua of the fields $\Phi_1^{\nu}, \Phi_2^{\nu}$ and $\Phi_{3'}^{\nu}$, whose alignments give rise to TB mixing in the neutrino sector, are driven by the fields Y_2^{ν} and $Z_{3'}^{\nu}$. The part of the superpotential responsible for the correct alignment reads

$$\begin{split} &a_{1}^{\nu}Y_{2}^{\nu}\Phi_{1}^{\nu}\Phi_{2}^{\nu}+a_{2}^{\nu}Y_{2}^{\nu}(\Phi_{2}^{\nu})^{2}+a_{3}^{\nu}Y_{2}^{\nu}(\Phi_{3'}^{\nu})^{2}+b_{1}^{\nu}Z_{3'}^{\nu}\Phi_{1}^{\nu}\Phi_{3'}^{\nu}+b_{2}^{\nu}Z_{3'}^{\nu}\Phi_{2}^{\nu}\Phi_{3'}^{\nu}+b_{3}^{\nu}Z_{3'}^{\nu}(\Phi_{3'}^{\nu})^{2} \qquad (4.6) \\ &=a_{1}^{\nu}\Phi_{1}^{\nu}\left[Y_{2,1}^{\nu}\Phi_{2,2}^{\nu}+Y_{2,2}^{\nu}\Phi_{2,1}^{\nu}\right]+a_{2}^{\nu}\left[Y_{2,1}^{\nu}(\Phi_{2,1}^{\nu})^{2}+Y_{2,2}^{\nu}(\Phi_{2,2}^{\nu})^{2}\right] \\ &+a_{3}^{\nu}\left[Y_{2,1}^{\nu}((\Phi_{3',3}^{\nu})^{2}+2\Phi_{3',1}^{\nu}\Phi_{3',2}^{\nu})+Y_{2,2}^{\nu}((\Phi_{3',2}^{\nu})^{2}+2\Phi_{3',1}^{\nu}\Phi_{3',3}^{\nu})\right] \\ &+b_{1}^{\nu}\Phi_{1}^{\nu}\left[Z_{3',1}^{\nu}\Phi_{3',1}^{\nu}+Z_{3',2}^{\nu}\Phi_{3',3}^{\nu}+Z_{3',3}^{\nu}\Phi_{3',2}^{\nu}\right] \\ &+b_{2}^{\nu}\left[Z_{3',1}^{\nu}(\Phi_{2,1}^{\nu}\Phi_{3',2}^{\nu}+\Phi_{2,2}^{\nu}\Phi_{3',3}^{\nu})+Z_{3',2}^{\nu}(\Phi_{2,1}^{\nu}\Phi_{3',1}^{\nu}+\Phi_{2,2}^{\nu}\Phi_{3',2}^{\nu})+Z_{3',3}^{\nu}(\Phi_{2,1}^{\nu}\Phi_{3',3}^{\nu}+\Phi_{2,2}^{\nu}\Phi_{3',1}^{\nu})\right] \\ &+b_{3}^{\nu}\left[Z_{3',1}^{\nu}((\Phi_{3',1}^{\nu})^{2}-\Phi_{3',2}^{\nu}\Phi_{3',3}^{\nu})+Z_{3',2}^{\nu}((\Phi_{3',2}^{\nu})^{2}-\Phi_{3',1}^{\nu}\Phi_{3',3}^{\nu})+Z_{3',3}^{\nu}((\Phi_{3',3}^{\nu})^{2}-\Phi_{3',1}^{\nu}\Phi_{3',3}^{\nu})\right], \end{split}$$

where the coefficients a_i^{ν} and b_i^{ν} are undetermined complex parameters whose absolute values are of order one. The F-terms of the components of Y_2^{ν} and $Z_{3'}^{\nu}$ vanish if the VEVs of Φ_1^{ν} , Φ_2^{ν} and $\Phi_{3'}^{\nu}$ take the following form⁹

$$\langle \Phi_{3'}^{\nu} \rangle = \varphi_{3'}^{\nu} \begin{pmatrix} \omega^{-p} \\ \omega^{p} \\ 1 \end{pmatrix}, \quad \langle \Phi_{2}^{\nu} \rangle = \varphi_{2}^{\nu} \begin{pmatrix} 1 \\ \omega^{p} \end{pmatrix}, \quad \langle \Phi_{1}^{\nu} \rangle = \varphi_{1}^{\nu}, \quad p = 0, \pm 1, \quad (4.7)$$

with the scales, $\varphi_{3'}^{\nu}$, φ_{2}^{ν} , φ_{1}^{ν} , of the VEVs related through

$$\varphi_2^{\nu} = -\frac{b_1^{\nu}}{2b_2^{\nu}}\omega^p \varphi_1^{\nu} , \qquad (\varphi_{3'}^{\nu})^2 = \frac{b_1^{\nu}}{6b_2^{\nu}a_3^{\nu}} \left(a_1^{\nu} - \frac{b_1^{\nu}a_2^{\nu}}{2b_2^{\nu}}\right) \omega^{2p} (\varphi_1^{\nu})^2 , \qquad (4.8)$$

and φ_1^{ν} remaining undetermined. Thus, without assuming any fine-tuning among the parameters a_i^{ν} and b_i^{ν} in the superpotential the VEVs are expected to be of the same order of magnitude,

$$\varphi_{3'}^{\nu} \sim \varphi_2^{\nu} \sim \varphi_1^{\nu} . \tag{4.9}$$

This is a favourable situation as it ensures that all entries of the right-handed neutrino mass matrix are naturally of similar order of magnitude so that a non-hierarchical light neutrino mass spectrum is generated. In Eq. (2.21) the alignment with p=0 is given. Finally, we note that the alignment of the flavons relevant to the neutrino sector at LO arises at $\mathcal{O}(\lambda^8)$. Thus, all combinations of flavons coupling to the driving fields Y_2^{ν} and $Z_{3'}^{\nu}$ which might give a contribution to the alignment of order $\gtrsim \lambda^8$ have to be absent.

Concerning the alignment of the VEV of $\widetilde{\Phi}_3^d$ we notice that for this purpose two driving fields are required, \overline{X}_1^d and $X_{1'}^{\nu d}$. First, one aligns $\langle \widetilde{\Phi}_3^d \rangle$ through the non-renormalisable

⁹To be precise, we explicitly exclude solutions in which the fields Φ_1^{ν} and Φ_2^{ν} acquire a VEV whereas $\Phi_{3'}^{\nu}$ does not.

operator

$$\frac{1}{M}\overline{X}_{1}^{d}\Phi_{2}^{d}\Phi_{3}^{d}\widetilde{\Phi}_{3}^{d} \qquad (4.10)$$

$$= \frac{1}{M}\overline{X}_{1}^{d} \left[\Phi_{2,1}^{d} (\Phi_{3,1}^{d}\widetilde{\Phi}_{3,2}^{d} + \Phi_{3,2}^{d}\widetilde{\Phi}_{3,1}^{d} + \Phi_{3,3}^{d}\widetilde{\Phi}_{3,3}^{d}) + \Phi_{2,2}^{d} (\Phi_{3,1}^{d}\widetilde{\Phi}_{3,3}^{d} + \Phi_{3,2}^{d}\widetilde{\Phi}_{3,2}^{d} + \Phi_{3,3}^{d}\widetilde{\Phi}_{3,1}^{d}) \right].$$

Using the alignment of $\langle \Phi_2^d \rangle$ and $\langle \Phi_3^d \rangle$ as discussed above we can immediately infer from setting the F-term of \overline{X}_1^d to zero that

$$\langle \widetilde{\Phi}_{3,1}^d \rangle = 0 , \qquad (4.11)$$

so that only the second and third entry of $\langle \widetilde{\Phi}_3^d \rangle$ can acquire a non-vanishing value. In order to correlate these entries we employ the field $X_{1'}^{\nu d}$ which couples $\widetilde{\Phi}_3^d$ to $\Phi_{3'}^{\nu}$ through

$$X_{1'}^{\nu d} \widetilde{\Phi}_{3}^{d} \Phi_{3'}^{\nu} = X_{1'}^{\nu d} (\widetilde{\Phi}_{3.1}^{d} \Phi_{3'.1}^{\nu} + \widetilde{\Phi}_{3.2}^{d} \Phi_{3'.3}^{\nu} + \widetilde{\Phi}_{3.3}^{d} \Phi_{3'.2}^{\nu}) . \tag{4.12}$$

For $\langle \Phi_{3'}^{\nu} \rangle$ being already aligned, the vanishing of the F-term of $X_{1'}^{\nu d}$

$$\langle \widetilde{\Phi}_{3,1}^d \rangle + \langle \widetilde{\Phi}_{3,2}^d \rangle + \langle \widetilde{\Phi}_{3,3}^d \rangle = 0 ,$$
 (4.13)

together with Eq. (4.11) shows that $\langle \widetilde{\Phi}_{3,2}^d \rangle$ and $\langle \widetilde{\Phi}_{3,3}^d \rangle$ have to be equal up to a relative sign. Thus, $\langle \widetilde{\Phi}_3^d \rangle$ is fully aligned as

$$\langle \widetilde{\Phi}_3^d \rangle \propto \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} . \tag{4.14}$$

Note that the operator from which $\langle \widetilde{\Phi}_{3,1}^d \rangle = 0$ is inferred arises at order λ^6 whereas the operator responsible for the equality of $\langle \widetilde{\Phi}_{3,2}^d \rangle$ and $\langle \widetilde{\Phi}_{3,3}^d \rangle$ is of order λ^7 , using the orders of magnitude shown in Eq. (2.13) and Eq. (2.25).

Finally, the vacua of the flavons, Φ_2^u and Φ_2^u , responsible for giving masses to up quarks at LO, can be aligned with the help of two driving fields Y_2^{du} and X_1^u . The field Y_2^{du} allows to couple the flavons Φ_2^u and Φ_2^d through the operator

$$Y_2^{du} \Phi_2^d \Phi_2^u = Y_{2,1}^{du} \Phi_{2,1}^d \Phi_{2,1}^u + Y_{2,2}^{du} \Phi_{2,2}^d \Phi_{2,2}^u . \tag{4.15}$$

Thus, from the vanishing of the F-term of $Y_{2,1}^{du}$ under the condition that $\langle \Phi_{2,1}^d \rangle \neq 0$ holds, as discussed above, we immediately find that

$$\langle \Phi_2^u \rangle \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{4.16}$$

Similarly, the field X_1^u couples the two fields Φ_2^u and $\widetilde{\Phi}_2^u$ through

$$X_1^u(\Phi_{2,1}^u\widetilde{\Phi}_{2,2}^u + \Phi_{2,2}^u\widetilde{\Phi}_{2,1}^u). (4.17)$$

Taking $\langle \Phi_2^u \rangle$ to be aligned as given in Eq. (4.16), we derive from the vanishing *F*-term of X_1^u that $\langle \tilde{\Phi}_2^u \rangle$ is aligned in the same way as $\langle \Phi_2^u \rangle$, i.e.

$$\langle \widetilde{\Phi}_2^u \rangle \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{4.18}$$

These alignments are induced by operators that arise - according to Eqs. (2.8,2.13) - at the order λ^5 and λ^8 , respectively.

Since some of the equations leading to the alignment of the flavon VEVs do not have a unique solution, we arrive at a total 24 different degenerate vacua (not counting the ones in which $\langle \Phi_{3'}^{\nu} \rangle = 0$). We note that these 24 sets are related by S_4 transformations. Choosing one of the sets different from the one presented in Eqs (2.6,2.10,2.21) clearly leads to fermion mass matrices which are of a different form from the one of those given in section 2. However, we have checked explicitly that all these sets of different fermion mass matrices are related to the one found in section 2 by S_4 transformations performed on the matter superfields T_3 , T, F and N. We emphasise that the results for fermion mixings are not changed by these transformations, because left-handed quarks as well as left-handed leptons transform in the same way. Thus, our choice of the vacuum structure is a convention that can be used without loss of generality.

Similar to the fact that the F-terms of the driving fields are the origin of the alignment of the flavon VEVs, we can derive from the F-terms of the latter fields the vacuum structure of the driving fields. As all terms in the flavon superpotential are linear in the driving fields, the configuration in which all these fields have vanishing VEVs is in any case a solution. However, in our model we find that, plugging in the alignment of the flavon VEVs, this is not the only possible solution satisfying the requirement that all F-terms of the flavons vanish. In principle, the two fields X_1^d and the second component of Y_2^{du} might have non-zero VEVs which fulfil a non-trivial relation. The absolute size of these VEVs is not determined, however their relative one. We note that non-vanishing VEVs for driving fields could induce a μ -term for H_5 and $H_{\overline{5}}$ which, in our model, is forbidden by the $U(1)_R$ symmetry. In the following we will, however, assume that all VEVs of the driving fields are zero.

4.2 Discussion of dangerous operators in the flavon superpotential

For specific choices of x, y and z additional operators which (can) spoil the above alignment might be allowed by the U(1) symmetry as well. Thus, it is necessary to check each of the 26 possible choices of U(1) charges x, y and z displayed in table 2 for such unwanted operators, using the vacuum alignments generated at the LO as shown in the preceding section. We classify all operators as unwanted which lead to contributions proportional to the same or to a lower power in λ than the LO terms given above.

As an example of an unsuccessful case which is excluded by our procedure, consider the U(1) charge assignment #1 with (x, y, z) = (1, 2, 5). In this case the operator

 $Z_{3'}^{\nu}\Phi_3^d(\Phi_2^d)^2/M$ is allowed by all the symmetries of the model. Inserting the desired vacuum structure we arrive at a contribution of the form $Z_{3',2}^{\nu}\varphi_3^d(\varphi_2^d)^2/M$ being of order λ^4 . This has to be compared to the terms given in Eq. (4.6) leading to the alignment of $\langle \Phi_{3',2,1}^{\nu} \rangle$ which are of order λ^8 . Thus, the additional operator $Z_{3'}^{\nu}\Phi_3^d(\Phi_2^d)^2/M$ gives a contribution dominating even the assumed LO one, so that the U(1) charge assignment #1 has to be discarded.

Eventually, we are left with four potentially successful U(1) charge assignments (x, y, z) for which we do not find any operators that strongly perturb the LO alignment if the flavons assume their LO VEVs. These are

$$#10: (4,5,2), #13: (5,4,1), #21: (3,-2,4), #25: (4,-1,5). (4.19)$$

We note that two of the solutions, namely #10 and #21, allow for operators which could in principle strongly perturb the LO result, $Y_2^d\Phi_2^d\Phi_2^u$, $X_1^u\Phi_2^d(\Phi_3^d)^2/M$ and $Y_2^d(\Phi_2^d)^3/M$, respectively. However, inserting the LO structure of the flavon VEVs we find that these operators give vanishing contributions. Nevertheless, they might still perturb the vacuum alignment if corrections to the LO vacua, caused by subleading terms, are taken into account (see below). In contrast to this, the solutions #13 and #25 do not allow for any operator which can strongly perturb the LO alignment, irrespective of the inserted vacua. Finally, we remark that for the choice #22 of U(1) charges, (x, y, z) = (3, -2, 5), there is one operator $M_V \overline{X}_1^d \Phi_1^{\nu}$ which, depending on the size of the mass scale M_V , might or might not spoil the vacuum alignment achieved at LO. Choosing $M_V \lesssim \lambda^3 M$ renders the associated contribution subdominant compared to the one coming from the LO term, displayed in Eq.(4.10). However, since we would like to avoid the presence of such additional mass scales in the flavon superpotential at this stage of the study, we discard case #22.

4.3 Correlations among the flavon VEVs

Having obtained the structure of the vacuum alignment, we now turn to the question of relating the scales of the flavon VEVs. So far, the only such relation is the one between the three flavons, relevant for right-handed neutrino masses, as stated in Eq. (4.9). Such a correlation of scales of more flavon VEVs can be achieved by adding further driving fields.

Referring to the detailed analysis in appendix D for the four viable choices of U(1) charges, #10, #13, #21 and #25, we find that only in case #13 is it possible to find two (independent) further relations among the flavon VEVs, if we introduce two further driving fields, transforming as singlets under S_4 . This result is achieved, if terms of a minimum size of order λ^9 are considered for scales of the flavon VEVs according to Eqs. (2.8,2.13,2.25), and the possibility of having couplings with positive mass dimension in the superpotential is not taken into account.

Explicitly we find

$$M\varphi_2^u \sim \varphi_2^d \widetilde{\varphi}_3^d$$
 and $M^2 \widetilde{\varphi}_2^u \widetilde{\varphi}_3^d \sim \varphi_2^d (\varphi_3^d)^3$, (4.20)

which comprise together with Eq. (4.9) the maximum set of correlations that we can achieve in the context of our 26 possible U(1) charge sets, see table 2. As a consequence, the eight flavon VEV scales face four constraints, thus leaving four parameters undetermined.

In the following we shall choose the flavon VEVs

$$\widetilde{\varphi}_2^u$$
, φ_3^d , φ_2^d and φ_1^{ν} , (4.21)

by hand to have the following orders

$$\widetilde{\varphi}_2^u/M \sim \lambda^4$$
, $\varphi_3^d/M \sim \lambda^2$, $\varphi_2^d/M \sim \lambda$ and $\varphi_1^{\nu}/M \sim \lambda^4$. (4.22)

Then, using the above correlations, we can deduce without further assumption

$$\varphi_2^u/M \sim \lambda^4$$
, $\widetilde{\varphi}_3^d/M \sim \lambda^3$, $\varphi_2^\nu/M \sim \lambda^4$ and $\varphi_{3'}^\nu/M \sim \lambda^4$. (4.23)

We find that the VEVs of the additional driving fields leading to the two further correlations have to vanish. This is required by the F-term equations of the flavons, if the LO alignments of Eqs. (2.6,2.10,2.21) are applied. As will be discussed in section 5.1 and in more detail in the second part of appendix D, the number of undetermined parameters among the flavon VEVs, see Eq. (4.21), can be further reduced if we allow for couplings with positive mass dimension in the flavon superpotential.

5 A specific model at NLO

Fixing the U(1) charges to take particular numerical values may allow certain operators that are forbidden for a general set (x,y,z) of U(1) charges so it is mandatory to study each model case by case. In this section we discuss the full results at NLO for the particularly promising model #13 where the U(1) charges are specified by (x,y,z)=(5,4,1). We note that we checked that the results of the study of the messenger sector, relevant in order to properly generate the two operators $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$ and $(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3H_{\overline{5}}/M^3$, are not altered by this choice of U(1) charges, especially no extra terms, not present in appendix B, arise (at the renormalisable level).

5.1 Flavon superpotential

We first summarise the operator structures arising at LO in the flavon superpotential

$$\begin{split} &X_{1}^{d}(\Phi_{2}^{d})^{2}+\frac{1}{M^{2}}Y_{2}^{d}(\Phi_{2}^{d})^{2}(\Phi_{3}^{d})^{2}\\ &+Y_{2}^{\nu}\Phi_{1}^{\nu}\Phi_{2}^{\nu}+Y_{2}^{\nu}(\Phi_{2}^{\nu})^{2}+Y_{2}^{\nu}(\Phi_{3'}^{\nu})^{2}+Z_{3'}^{\nu}\Phi_{1}^{\nu}\Phi_{3'}^{\nu}+Z_{3'}^{\nu}\Phi_{2}^{\nu}\Phi_{3'}^{\nu}+Z_{3'}^{\nu}(\Phi_{3'}^{\nu})^{2}\\ &+\frac{1}{M}\overline{X}_{1}^{d}\Phi_{2}^{d}\Phi_{3}^{d}\widetilde{\Phi}_{3}^{d}+X_{1'}^{\nu d}\widetilde{\Phi}_{3}^{d}\Phi_{3'}^{\nu}+Y_{2}^{du}\Phi_{2}^{d}\Phi_{2}^{u}+X_{1}^{u}\Phi_{2}^{u}\widetilde{\Phi}_{2}^{u}\\ &+\frac{1}{M}X_{1}^{\text{new}}\Phi_{2}^{u}(\Phi_{3}^{d})^{2}+\frac{1}{M^{2}}X_{1}^{\text{new}}\Phi_{2}^{d}\widetilde{\Phi}_{3}^{d}(\Phi_{3}^{d})^{2}+\frac{1}{M}\widetilde{X}_{1'}^{\text{new}}\widetilde{\Phi}_{2}^{u}\Phi_{3}^{d}\widetilde{\Phi}_{3}^{d}+\frac{1}{M^{3}}\widetilde{X}_{1'}^{\text{new}}\Phi_{2}^{d}(\Phi_{3}^{d})^{4}. \end{split}$$

The effect of these operators has been discussed in detail in the preceding section and in appendix D.

In addition, for case #13, we find several operators which are subleading in the expansion in λ relative to these, when the orders of the flavon VEVs are chosen as in section 2 in order to reproduce in a satisfying way all fermion masses and mixings. These subleading operators in general perturb the result for the vacuum alignment at the LO in a particular way and thus affect the results for fermion masses and mixings.

In the following we consider all subleading operators which can contribute at a level of up to and including order λ^{12} , for scales of VEVs as shown in Eqs. (2.8,2.13,2.25). Due to the four undetermined and the four fixed VEV scales, see Eqs. (4.21,4.23), we have to parametrise the perturbed vacua of the flavons in the following way

$$\langle \Phi_2^u \rangle = \begin{pmatrix} \Delta_{2,1}^u \\ \varphi_2^u + \Delta_{2,2}^u \end{pmatrix}, \qquad \langle \widetilde{\Phi}_2^u \rangle = \begin{pmatrix} \widetilde{\Delta}_{2,1}^u \\ \widetilde{\varphi}_2^u \end{pmatrix}, \qquad (5.2)$$

$$\langle \Phi_3^d \rangle = \begin{pmatrix} \Delta_{3,1}^d \\ \varphi_3^d \\ \Delta_{3,3}^d \end{pmatrix}, \qquad \langle \widetilde{\Phi}_3^d \rangle = \begin{pmatrix} \widetilde{\Delta}_{3,1}^d \\ -(\widetilde{\varphi}_3^d + \widetilde{\Delta}_{3,2}^d) \\ \widetilde{\varphi}_3^d + \widetilde{\Delta}_{3,3}^d \end{pmatrix}, \qquad \langle \Phi_2^d \rangle = \begin{pmatrix} \varphi_2^d \\ \Delta_{2,2}^d \end{pmatrix},$$

$$\langle \Phi_{3'}^\nu \rangle = \begin{pmatrix} \varphi_{3'}^\nu + \Delta_{3',1}^\nu \\ \varphi_{3'}^\nu + \Delta_{3',2}^\nu \\ \varphi_{3'}^\nu + \Delta_{3',3}^\nu \end{pmatrix}, \qquad \langle \Phi_2^\nu \rangle = \begin{pmatrix} \varphi_2^\nu + \Delta_{2,1}^\nu \\ \varphi_2^\nu + \Delta_{2,2}^\nu \end{pmatrix} \quad \text{and} \quad \langle \Phi_1^\nu \rangle = \varphi_1^\nu.$$

Including all the above leading and subleading operators, we solve the equations originating from the F-terms of the driving fields order by order in λ , up to and including λ^{12} in order to determine the size of all the shifts $\Delta^f_{i,j}$, $f = u, d, \nu$. We find as result that the shifts are of the order in λ

$$\begin{split} & \Delta^{u}_{2,1}/M = \delta^{u}_{2,1}\lambda^{8} \;, \;\; \Delta^{u}_{2,2}/M = \delta^{u}_{2,2}\lambda^{6} \;, \;\; \widetilde{\Delta}^{u}_{2,1}/M = \widetilde{\delta}^{u}_{2,1}\lambda^{6} \;, \;\; \Delta^{d}_{3,1}/M = \delta^{d}_{3,1}\lambda^{6} \;, \quad (5.3) \\ & \Delta^{d}_{3,3}/M = \delta^{d}_{3,3}\lambda^{6} \;, \;\; \widetilde{\Delta}^{d}_{3,1}/M = \widetilde{\delta}^{d}_{3,1}\lambda^{7} \;, \;\; \widetilde{\Delta}^{d}_{3,2}/M = \widetilde{\delta}^{d}_{3,2}\lambda^{5} \;, \;\; \widetilde{\Delta}^{d}_{3,3}/M = \widetilde{\delta}^{d}_{3,3}\lambda^{5} \;, \\ & \Delta^{d}_{2,2}/M = \delta^{d}_{2,2}\lambda^{7} \;, \;\; \Delta^{\nu}_{3',1}/M = \delta^{\nu}_{3',1}\lambda^{8} \;, \;\; \Delta^{\nu}_{3',2}/M = \delta^{\nu}_{3',2}\lambda^{8} \;, \;\; \Delta^{\nu}_{3',3}/M = \delta^{\nu}_{3',3}\lambda^{8} \;, \\ & \Delta^{\nu}_{2,1}/M = \delta^{\nu}_{2,1}\lambda^{8} \;\; \text{and} \;\; \Delta^{\nu}_{2,2}/M = \delta^{\nu}_{2,2}\lambda^{8} \;, \end{split}$$

where $\delta^f_{i,j}$, $f=u,d,\nu$, are complex numbers with absolute value of order one, determined by the couplings of the superpotential. Notice that the shifts associated with the components of the flavon $\Phi^{\nu}_{3'}$ are equal at this level, i.e.

$$\Delta_{3',1}^{\nu} = \Delta_{3',2}^{\nu} = \Delta_{3',3}^{\nu} , \qquad (5.4)$$

so that the alignment, achieved at LO, is not perturbed up to the level λ^8 . Since we are not interested in the actual relation between $\varphi^{\nu}_{3'}$ and φ^{ν}_{1} , we can absorb the shifts of the VEVs of the components of the field $\Phi^{\nu}_{3'}$ into the LO VEV $\varphi^{\nu}_{3'}$. As we will see in section 5.2, this leads to the fact that tri-maximal mixing remains still preserved in the neutrino sector.

As one can see all ratios $\Delta_{i,j}^f/\varphi_i^f$ are small, at most of order λ^2 , so that the shifts relative to the LO alignment are small. Nevertheless these might lead to relevant corrections to LO results for fermion masses and mixings, which is discussed in section 5.2. Using the parametrisation in Eq. (5.2) and the results of the shifts given in Eq. (5.3), the *F*-terms of all 15 driving fields vanish up to order λ^{12} , apart from the one associated with the field $Y_{2,2}^{du}$ which has a contribution at order λ^{11} which can only vanish if either one coupling of the flavon superpotential is tuned to cancel the term or one of the involved flavon VEVs vanishes. This tuning can be understood because the *F*-term equations of the 15 driving fields have to be fulfilled by solving for the 14 shifts $\Delta_{i,j}^f$. However, since the required tuning arises only at order λ^{11} it has to be considered only a minor drawback in the construction of the flavon superpotential.

Finally, we briefly comment on how to ensure the spontaneous breaking of the family symmetry (i.e. avoid the trivial solution with all flavon VEVs being zero) by introducing a coupling with mass dimension two in the superpotential. This can be achieved by adding a driving field $V_0 \sim (1,0)$ which is a total singlet under $S_4 \times U(1)$ so that the term $M_{V_0}^2 V_0$ is allowed. At the same time, a combination of the undetermined VEVs, $\widetilde{\varphi}_2^u$, φ_3^d , φ_2^d and φ_1^ν , see Eq. (4.21), becomes fixed through M_{V_0} . Furthermore, we find that introducing another driving field $V_2 \sim (2, -8)$ gives rise to a term $M_{V_2} V_2 \Phi_2^\nu$. Considering operators resulting in contributions of order λ^8 or larger, the field V_2 leads to two additional constraints on the undetermined VEVs so that only one free parameter remains. See second part of appendix D for details.

We remark that one could fix the remaining undetermined parameter among the flavon VEVs through a Fayet-Iliopoulos term of an appropriate size provided the U(1) symmetry is gauged. However, we do not pursue this possibility further.

5.2 Fermion masses and mixings

In the following we study the effects of the subleading operators. We include corrections caused by the shifted vacua as given above and the allowed multi-flavon insertions with up to eight flavons.

5.2.1 Quark sector

Including terms up to order λ^8 we find that the up quark mass matrix remains nearly diagonal apart from the off-diagonal elements (23) and (32) which are of order λ^7 . This correction originates from the operator structure $TT_3(\Phi_2^d)^3(\Phi_3^d)^2H_5/M^5$. The diagonal elements get corrected compared to the LO result: we find that the (11) element does not only arise from the LO operator $TT\Phi_2^u\widetilde{\Phi}_2^uH_5/M^2$ but also from the operator $TT\Phi_2^uH_5/M$ if the non-zero shift $\Delta_{2,1}^u$ is taken into account. However, we include the latter contribution into the former one. The (22) element receives a correction of order λ^6 stemming from the insertion of the shifted vacuum of Φ_2^u into the LO operator $TT\Phi_2^uH_5/M$. All corrections which might arise to the (33) element can be absorbed into the coupling of the LO tree-level operator $T_3T_3H_5$. After taking into account possible re-phasing of the right-handed

fermion fields we find that M_u can be parametrised as

$$M_{u} = \begin{pmatrix} y_{u}\lambda^{8} & 0 & 0\\ 0 & y_{c}\lambda^{4} & 0\\ 0 & 0 & y_{t} \end{pmatrix} v_{u} + \begin{pmatrix} 0 & 0 & 0\\ 0 & z_{1}^{u}e^{-i\alpha_{u,1}}\lambda^{6} & z_{2}^{u}e^{-i\alpha_{u,2}}\lambda^{7}\\ 0 & z_{2}^{u}e^{-i\alpha_{u,3}}\lambda^{7} & 0 \end{pmatrix} v_{u} .$$
 (5.5)

Note that the parameters $y_{u,c,t}$ and $z_{1,2}^u$ are real and positive and the phases $\alpha_{u,i}$ are between 0 and 2π . Here and in the following we display each mass matrix as the sum of the LO and the NLO result. For the up quark masses we find

$$m_u = y_u \lambda^8 v_u , \quad m_c = (y_c \lambda^4 + \mathcal{O}(\lambda^6)) v_u , \quad m_t = (y_t + \mathcal{O}(\lambda^{14})) v_u .$$
 (5.6)

Thus, all corrections coming from NLO terms are small. In particular, all mixing angles in the up quark sector are negligible.

Similarly, we find the following parametrisation for the down quark mass matrix

$$M_{d} = \begin{pmatrix} 0 & \widetilde{x}_{2}\lambda^{5} & -\widetilde{x}_{2}e^{i\alpha_{d,2}}\lambda^{5} \\ -\widetilde{x}_{2}\lambda^{5} & y_{s}e^{-i\alpha_{d,1}}\lambda^{4} & (-y_{s}e^{-i(\alpha_{d,1}-\alpha_{d,2})}\lambda^{4} + \widetilde{x}_{2}e^{i\alpha_{d,2}}\lambda^{5}) \\ 0 & 0 & y_{b}\lambda^{2} \end{pmatrix} v_{d}$$

$$+ \begin{pmatrix} z_{1}^{d}e^{i\psi_{d,1}}\lambda^{8} & -z_{6}^{d}e^{i\psi_{d,6}}\lambda^{7} & 0 \\ 0 & z_{4}^{d}e^{i\psi_{d,4}}\lambda^{6} & -z_{5}^{d}e^{i\psi_{d,5}}\lambda^{6} \\ z_{3}^{d}e^{i\psi_{d,3}}\lambda^{6} & z_{2}^{d}e^{i\psi_{d,2}}\lambda^{6} & 0 \end{pmatrix} v_{d} ,$$

$$(5.7)$$

where we have only displayed the first subleading contribution to each of the different matrix elements up to order λ^8 . The parameters y_s and y_b are associated with the LO operators $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$ and $FT_3\Phi_3^dH_{\overline{5}}/M$, respectively. \widetilde{x}_2 coincides with the LO parameter x_2 as given in Eq. (2.4), up to corrections of order λ^2 which are due to the shift $\widetilde{\Delta}_{3,2}^d$. The (11) element is of order λ^8 and originates from several possible contractions of the operator structures $FT\widetilde{\Phi}_2^u\Phi_2^d\widetilde{\Phi}_3^dH_{\overline{45}}/M^3$ and $FT(\Phi_2^d)^2(\Phi_3^d)^3H_{\overline{45}}/M^5$. The (32) element arises at order λ^6 from the following two sources: through plugging the shifted vacuum of Φ_3^d in the LO operator $FT_3\Phi_3^dH_{\overline{5}}/M$ and through the subleading operator $FT_3\widetilde{\Phi}_2^u\Phi_3^dH_{\overline{5}}/M^2$. Similarly, the (31) element of order λ^6 arises from the LO term $FT_3\Phi_3^dH_{\overline{5}}/M$ and is proportional to the shift $\Delta_{3,1}^d$. The corrections of order λ^6 in the (22) and the (23) elements, encoded in the parameters z_4^d and z_5^d , originate from the LO operator $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$ if the shifts of the vacuum alignment are included, and are proportional to $\widetilde{\Delta}_{3,3}^d$ and to $\widetilde{\Delta}_{3,2}^d$, respectively. Finally, the correction to the (12) element is again the result of the shifted vacuum of $\widetilde{\Phi}_3^d$, this time plugged into $(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3H_{\overline{5}}/M^3$ and is generically of order λ^7 . We note that also in case of M_d (and M_e , see below) all parameters, $y_{b,s}$, \tilde{x}_2 and z_i^d , are real and positive and that all appearing phases, $\alpha_{d,i}$ and $\psi_{d,i}$, are within the interval $[0,2\pi)$. The mass matrix in Eq. (5.7) leads to down quark masses of the form

$$m_d = \left(\frac{\widetilde{x}_2^2}{y_s}\lambda^6 + \mathcal{O}(\lambda^8)\right)v_d, \quad m_s = \left(y_s\lambda^4 + \mathcal{O}(\lambda^6)\right)v_d, \quad m_b = \left(y_b\lambda^2 + \mathcal{O}(\lambda^6)\right)v_d. \quad (5.8)$$

For the quark mixing angles we find

$$\sin \theta_{13}^q = \frac{\widetilde{x}_2}{y_b} \lambda^3 , \quad \tan \theta_{12}^q = \frac{\widetilde{x}_2}{y_s} \lambda + \mathcal{O}(\lambda^3) \quad \text{and} \quad \tan \theta_{23}^q = \frac{y_s}{y_b} \lambda^2 + \mathcal{O}(\lambda^3) , \tag{5.9}$$

showing that the angles θ_{ij}^q are only determined by the LO results and all subleading corrections are very small. The calculation of the Jarlskog invariant J_{CP} yields

$$J_{CP} = \frac{\widetilde{x}_2^3}{y_s y_b^2} \lambda^7 \sin \alpha_{d,1} + \mathcal{O}(\lambda^8) , \qquad (5.10)$$

which turns out to be slightly below its expected size of λ^6 [37]. Eqs. (5.8,5.9) confirm the achievement of the GST relation [29] in our model, even after including corrections,

$$\tan \theta_{12}^q \approx \sqrt{\frac{m_d}{m_s}} \ . \tag{5.11}$$

Due to the fact that only the parameters associated with the LO contributions are relevant for the determination of masses, mixing angles and CP violation, the model might turn out to be incapable of fitting the precise values for the quantities determined from experiments [37]. However, we point out that in our model these quantities are evaluated at a high energy scale and any renormalisation group and threshold effects [38], which among other things depend also on the actual value of $\tan \beta$, are not taken into account in this analysis.

5.2.2 Lepton sector

Coming to the lepton sector, we first note that the structure of the charged lepton mass matrix is analogous to the one of M_d , apart from the GJ factor and the slightly different positions of the phases, after re-phasing of all right-handed fields,

$$M_{e} = \begin{pmatrix} 0 & -\tilde{x}_{2}\lambda^{5} & 0\\ \tilde{x}_{2}\lambda^{5} & -3y_{s}e^{-i\alpha_{d,1}}\lambda^{4} & 0\\ -\tilde{x}_{2}\lambda^{5} & (3y_{s}e^{-i\alpha_{d,1}}\lambda^{4} + \tilde{x}_{2}\lambda^{5}) & y_{b}\lambda^{2} \end{pmatrix} v_{d}$$

$$+ \begin{pmatrix} -3z_{1}^{d}e^{i\psi_{d,1}}\lambda^{8} & 0 & z_{3}^{d}e^{i(\alpha_{d,2}+\psi_{d,3})}\lambda^{6}\\ -z_{6}^{d}e^{i\psi_{d,6}}\lambda^{7} & -3z_{4}^{d}e^{i\psi_{d,4}}\lambda^{6} & z_{2}^{d}e^{i(\alpha_{d,2}+\psi_{d,2})}\lambda^{6}\\ 0 & 3z_{5}^{d}e^{-i(\alpha_{d,2}-\psi_{d,5})}\lambda^{6} & 0 \end{pmatrix} v_{d}.$$

$$(5.12)$$

The mass matrix given in Eq. (5.12) leads to charged lepton masses

$$m_e = \left(\frac{\widetilde{x}_2^2}{3y_s}\lambda^6 + \mathcal{O}(\lambda^8)\right)v_d, \quad m_\mu = \left(3y_s\lambda^4 + \mathcal{O}(\lambda^6)\right)v_d, \quad m_\tau = \left(y_b\lambda^2 + \mathcal{O}(\lambda^6)\right)v_d,$$

$$(5.13)$$

which, similar to the quark masses, up to small corrections, are only determined by the LO terms. The GJ relations [30] are confirmed by Eqs. (5.8) and (5.13). The charged

lepton mixing angles are of the form

$$\sin \theta_{13}^e = \frac{z_3^d}{y_b} \lambda^4 + \mathcal{O}(\lambda^5) , \quad \tan \theta_{12}^e = \frac{\widetilde{x}_2}{3y_s} \lambda + \mathcal{O}(\lambda^3) , \qquad (5.14)$$

$$\tan \theta_{23}^e = \left| 9 \left(\frac{y_s}{y_b} \right)^2 - e^{i(\alpha_{d,2} + \psi_{d,2})} \left(\frac{z_2^d}{y_b} \right) \right| \lambda^4 + \mathcal{O}(\lambda^5) ,$$

coinciding with the estimate found in section 2.

The Dirac neutrino mass matrix elements also receive small corrections of order λ^4 and λ^6 , respectively. They read

$$M_D = \begin{pmatrix} y_D & 0 & 0 \\ 0 & 0 & y_D \\ 0 & y_D & 0 \end{pmatrix} v_u + \begin{pmatrix} 2z_3^D \lambda^6 & z_2^D \lambda^6 & z_1^D \lambda^4 \\ z_2^D \lambda^6 & z_1^D \lambda^4 & -(z_3^D - z_4^D) \lambda^6 \\ z_1^D \lambda^4 & -(z_3^D + z_4^D) \lambda^6 & z_2^D \lambda^6 \end{pmatrix} v_u , \quad (5.15)$$

with y_D being the coupling accompanying the LO tree-level term FNH_5 . The corrections associated with the two parameters z_1^D and z_2^D originate from the operator structure $FN\widetilde{\Phi}_2^uH_5/M$. The one associated with z_2^D is additionally suppressed by a factor of λ^2 because it is not proportional to the VEV $\widetilde{\varphi}_2^u$ but rather to the shift $\widetilde{\Delta}_{2,1}^u$. The source of the two corrections to the (11), (23) and (32) elements of order λ^6 , encoded in z_3^D and z_4^D , is the operator structure $FN(\Phi_2^d)^4\Phi_3^dH_5/M^5$. The two different parameters refer to two different possible contractions of the operator.

In the right-handed neutrino mass matrix corrections are of a relative order λ^4 . They are encoded in two parameters, denoted by Z_1 and Z_2 in the following. The general form of M_R can be written as

$$M_{R} = \begin{pmatrix} A + 2C & B - C & B - C \\ B - C & B + 2C & A - C \\ B - C & A - C & B + 2C \end{pmatrix} \lambda^{4} M + \begin{pmatrix} 0 & Z_{1} & Z_{2} \\ Z_{1} & Z_{2} & 0 \\ Z_{2} & 0 & Z_{1} \end{pmatrix} \lambda^{8} M .$$
 (5.16)

In the above equation we have also introduced the parameters A, B and C, which (dominantly) originate from the LO terms $\alpha NN\Phi_1^{\nu}$, $\beta NN\Phi_2^{\nu}$, $\gamma NN\Phi_{3'}^{\nu}$, respectively. Note that at the same time some of the subleading contributions are absorbed by re-defining A as well as C. The latter incorporates then also the contribution coming from the operator $NN\tilde{\Phi}_2^u\Phi_{3'}^{\nu}/M$. One source of the contributions, parametrised by Z_1 and Z_2 , are the shifts $\Delta_{2,1}^{\nu}$ and $\Delta_{2,2}^{\nu}$ if the LO term $NN\Phi_2^{\nu}$ is evaluated with the shifted vacuum of Φ_2^{ν} . Apart from that, the subleading term $NN\Phi_2^{\nu}\tilde{\Phi}_2^{u}/M$ contributes to the correction associated with Z_1 , while the two operator structures $NN\Phi_1^{\nu}\tilde{\Phi}_2^{u}/M$ and $NN(\Phi_2^d)^8/M^7$ give a contribution to the (13) and (22) elements of M_R . The effective light neutrino mass matrix which arises from the type I see-saw mechanism can be arranged as

$$m_{\nu}^{eff} = \begin{pmatrix} B_{\nu} + C_{\nu} - A_{\nu} & A_{\nu} & A_{\nu} \\ A_{\nu} & B_{\nu} & C_{\nu} \\ A_{\nu} & C_{\nu} & B_{\nu} \end{pmatrix} \begin{pmatrix} v_{u}^{2} \\ \lambda^{4}M \end{pmatrix}$$

$$+ \begin{pmatrix} z_{1}^{\nu} & z_{2}^{\nu} & z_{3}^{\nu} \\ z_{2}^{\nu} & z_{1}^{\nu} + z_{3}^{\nu} - z_{4}^{\nu} & z_{4}^{\nu} \\ z_{3}^{\nu} & z_{4}^{\nu} & z_{1}^{\nu} + z_{2}^{\nu} - z_{4}^{\nu} \end{pmatrix} \begin{pmatrix} v_{u}^{2} \\ M \end{pmatrix} .$$

$$(5.17)$$

 A_{ν} , B_{ν} and C_{ν} parametrise the LO contributions and the four independent parameters z_i^{ν} the corrections to the light neutrino mass matrix. In the following we will assume all these parameters to be real since there is no experimental evidence for CP violating phases in the lepton sector yet. As expected all corrections to the light neutrino masses arise at a relative level of λ^4 . Due to the fact that, up to the order λ^8 , the shifted vacuum of the flavon $\Phi_{3'}^{\nu}$ reveals the same alignment as the LO one, the tri-maximally mixed state remains an eigenstate of the light neutrino mass matrix m_{ν}^{eff} (as well as of the right-handed neutrino mass matrix M_R). The neutrino mixing angles are thus still given by the TB mixing values up to corrections of $\mathcal{O}(\lambda^4)$. Eventually, we find for the lepton mixing angles

$$\sin \theta_{13}^{l} = \frac{\widetilde{x}_{2}}{3\sqrt{2}y_{s}} \lambda + \mathcal{O}(\lambda^{3}) , \quad \sin^{2} \theta_{12}^{l} = \frac{1}{3} - \frac{2\widetilde{x}_{2}}{9y_{s}} \lambda \cos \alpha_{d,1} + \mathcal{O}(\lambda^{2}) , \qquad (5.18)$$

$$\sin^{2} \theta_{23}^{l} = \frac{1}{2} - \frac{\widetilde{x}_{2}^{2}}{36y_{s}^{2}} \lambda^{2} + \mathcal{O}(\lambda^{4}) ,$$

coinciding with the estimates given in Eq. (2.26). Comparing the results for quark and lepton mixing angles, Eqs. (5.9) and (5.18), we see that our model incorporates the correlations [31]

$$\sin^2 \theta_{12}^l \approx \frac{1}{3} - \frac{2}{9} \tan \theta_{12}^q \cos \alpha_{d,1} , \qquad (5.19)$$

and

$$\sin \theta_{13}^l \approx \tan \theta_{12}^q / (3\sqrt{2}) ,$$
 (5.20)

with $\alpha_{d,1}$ playing the role of the Dirac CP phase δ^l in the lepton sector, up to π

$$\delta^l = \alpha_{d,1} + \pi \ . \tag{5.21}$$

This relation holds up to corrections of order λ . Thus, we can write Eq. (5.19) also as

$$\sin^2 \theta_{12}^l \approx \frac{1}{3} \left(1 + 2\sqrt{2} \sin \theta_{13}^l \cos \delta^l \right) . \tag{5.22}$$

Note that Eq. (5.22) holds without loss of generality, although we have assumed all parameters in the neutrino sector to be real since, as shown in [31], the validity of this relation only depends on the fact that θ_{13}^e , θ_{23}^e and θ_{13}^ν are (much) smaller than θ_{12}^e .

It is convenient to define [39],

$$\sin \theta_{13}^l = \frac{r}{\sqrt{2}}, \quad \sin \theta_{12}^l = \frac{1}{\sqrt{3}}(1+s), \quad \sin \theta_{23}^l = \frac{1}{\sqrt{2}}(1+a),$$
 (5.23)

where we have introduced the three real parameters r, s, a to describe the deviations of the reactor, solar and atmospheric mixing angles from their TB values. The present model predicts these deviation parameters to be,

$$s \approx r \cos \delta^l, \quad r \approx \lambda/3, \quad a \approx -\lambda^2/36,$$
 (5.24)

up to renormalisation group effects and corrections associated with non-canonically normalised kinetic terms. While the first equation above is the usual sum rule in terms of deviation parameters [39], we emphasise that the model predicts another new relation

$$a \approx -r^2/4,\tag{5.25}$$

with $r \approx \lambda/3$, valid at the GUT scale.

In summary, all NLO corrections turn out to have a negligible effect on the results for fermion masses and mixings achieved at LO and presented in section 2.

6 Conclusions

In this article we have constructed a model of fermion masses and mixings based on the combination of the minimal GUT SU(5) and the family symmetry S_4 . The latter is also minimal in the sense that it is the smallest non-Abelian finite group which contains all the symmetries necessary to enforce TB neutrino mixing. At LO, the effective light neutrino mass matrix arises from the type I see-saw mechanism where the TB mixing structure is imprinted in the form of the Majorana mass matrix of the right-handed neutrinos. The latter in turn originates from the vacuum alignment of three different flavon fields $\Phi_{3'}^{\nu}$, $\Phi_{2'}^{\nu}$, Φ_{1}^{ν} . As the right-handed neutrino mass matrix contains three independent parameters, our model can accommodate all patterns for the neutrino masses; in particular we do not encounter the constraint of a neutrino mass sum rule as in the corresponding A_4 models. At the same time the TB neutrino mixing is independent of the particular values of the neutrino masses and also stable under inclusion of NLO corrections. Taking into account the corrections to the flavon alignments as well as higher-dimensional operators, we find that TB neutrino mixing remains exact up to $\mathcal{O}(\lambda^4) \sim 0.1\%$ at the GUT scale.

Regarding the masses of the charged fermions, we invoke additional SU(5) singlet flavon fields. Their LO alignments give rise to acceptable quark and charged lepton mass matrices, including the phenomenologically successful GJ and GST relations. The latter cannot be achieved in a generic effective theory, but require some specific set of messenger fields. Such a set has been explicitly constructed. Having introduced eight flavon fields, it is necessary to study all allowed superpotential operators with two matter fields and an arbitrary number of flavons. In order to forbid those terms which would spoil the LO results for the mass matrices, we introduce a new U(1) symmetry, parametrised by three integers (x, y, z). Their specific values are determined when discussing how to obtain the required vacuum alignment. In our model this originates from the Fterms of an additional set of fields, the driving fields, which cannot couple directly to the matter superfields. Solving the F-term equations of the driving fields in the SUSY limit, we can obtain the desired flavon alignments at LO. Additional driving fields are then added to obtain further correlations between the scales of the flavon VEVs. This study fixes our preferred choice of U(1) charges, given by (x, y, z) = (5, 4, 1). The number of undetermined parameters among the flavon VEVs can be minimised by considering driving fields allowing for couplings with positive mass dimension in the flavon superpotential. In

this way we can obtain by the (ad hoc) choice of the magnitude of two mass parameters and one flavon VEV, which remains undetermined, the correct size of the VEVs of all flavons coupling to the superfields T_3 , T, F and N, as required to achieve the observed fermion mass and mixing patterns in the quark and lepton sector. In the final part of this work we have scrutinised the NLO effects on the flavon alignments as well as the fermion mass matrices. Our results reveal that the NLO corrections have a negligible effect on quark and lepton masses and mixings, thus confirming the stability of the original LO structure of the model. Since the main purpose of this work is the study of fermion masses and mixings, we have not discussed the GUT Higgs sector and the corresponding (super-)potential necessary in order to correctly break SU(5) to the SM gauge group.

In conclusion we have constructed a SUSY GUT of Flavour based on $S_4 \times SU(5)$, together with an additional (global or local) Abelian symmetry, and studied it to NLO accuracy. We have specified the complete effective theory for general U(1) charges, valid just below the GUT scale, relevant for fermion masses and mixings, and performed a full operator analysis taking into account all relevant higher order terms with several insertions of flavons. The model includes a successful description of quark masses and mixing angles at LO incorporating the GST relation. In addition, at LO, charged lepton and down quark masses fulfil GJ relations. Our predictions apply just below the GUT scale, and the determination of the fermion masses and mixings at the electroweak scale would require a detailed investigation of renormalisation group and threshold effects which is beyond the scope of this paper. We have studied the vacuum alignment arising from F-terms to NLO and the resulting corrections have been shown to not affect the LO predictions significantly for specific choices of U(1) charges. A specific model evaluated to NLO predicts TB mixing in the neutrino sector very accurately up to corrections of order 0.1%. Including charged lepton mixing corrections leads to small deviations from TB lepton mixing described by a precise sum rule, with accurately maximal atmospheric mixing and a reactor mixing angle close to three degrees.

Acknowledgments

We thank Ferruccio Feruglio, Marco Serone and Robert Ziegler for discussions. CH thanks the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support during the completion of this work. SFK and CL acknowledge support from the STFC Rolling Grant ST/G000557/1. SFK is grateful to the Royal Society for a Leverhulme Trust Senior Research Fellowship.

Appendix

A Group theory of S_4

The group S_4 is the permutation group of four distinct objects and is isomorphic to the symmetry group O of a regular octahedron. Its order is 24 and it contains five real irreducible representations: 1, 1', 2, 3 and 3'. Only the two triplet representations are faithful. A decisive feature among the two triplets 3 and 3' is that only 3 can be identified with the fundamental representation of the continuous groups SO(3) and SU(3). The three generators S, T and U are of the following form for the five different representations

with $\omega = e^{2\pi i/3}$.

The generators fulfil the relations

$$\begin{split} S^2 &= \mathbb{1} \ , \ T^3 &= \mathbb{1} \ , \ U^2 &= \mathbb{1} \ , \\ (ST)^3 &= \mathbb{1} \ , \ (SU)^2 &= \mathbb{1} \ , \ (TU)^2 &= \mathbb{1} \ , \ (STU)^4 &= \mathbb{1} \ . \end{split}$$

Note that the minimal number of generators necessary to define S_4 is actually only two, compare e.g. [16]. However, in order to emphasise the correlation between the groups A_4 and S_4 it is advantageous to choose the set S, T and U, since then one easily sees that S and T alone generate the group A_4 , see fifth reference in [6]. Notice that similarly, the two generators T and U alone generate the group S_3 [40]. The character table is given in table 4. The Kronecker products are of the form

$$\begin{split} \mathbf{1} \times \mu &= \mu \;\; \forall \;\; \mu \;, \;\; \mathbf{1'} \times \mathbf{1'} = \mathbf{1} \;, \;\; \mathbf{1'} \times \mathbf{2} = \mathbf{2} \;, \\ \mathbf{1'} \times \mathbf{3} &= \mathbf{3'} \;, \;\; \mathbf{1'} \times \mathbf{3'} = \mathbf{3} \;, \\ \mathbf{2} \times \mathbf{2} &= \mathbf{1} + \mathbf{1'} + \mathbf{2} \;, \;\; \mathbf{2} \times \mathbf{3} = \mathbf{2} \times \mathbf{3'} = \mathbf{3} + \mathbf{3'} \;, \\ \mathbf{3} \times \mathbf{3} &= \mathbf{3'} \times \mathbf{3'} = \mathbf{1} + \mathbf{2} + \mathbf{3} + \mathbf{3'} \;, \;\; \mathbf{3} \times \mathbf{3'} = \mathbf{1'} + \mathbf{2} + \mathbf{3} + \mathbf{3'} \;. \end{split}$$

In the following we list the Clebsch-Gordan coefficients using the notation $a \sim \mathbf{1}$, $a' \sim \mathbf{1}'$, $(b_1, b_2)^t$, $(\tilde{b}_1, \tilde{b}_2)^t \sim \mathbf{2}$, $(c_1, c_2, c_3)^t$, $(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)^t \sim \mathbf{3}$, $(c_1', c_2', c_3')^t$, $(\tilde{c}_1', \tilde{c}_2', \tilde{c}_3')^t \sim \mathbf{3}'$.

			Clas	ses	
	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5
G	1	S	U	T	STU
n_i	1	3	6	8	6
h_i	1	2	2	3	4
1	1	1	1	1	1
1'	1	1	-1	1	-1
2	2	2	0	-1	0
3	3	-1	-1	0	1
3′	3	-1	1	0	-1

Table 4: Character table of the group S_4 . C_i denote the five classes of S_4 , n_i the number of distinct elements in the classes C_i and h_i the order of the elements contained in class C_i . For each of the classes we give a representative G in terms of the generators S, T and U.

For a singlet multiplied with a doublet or a triplet

$$\mathbf{1}^{(\prime)} \times \mathbf{2} : \qquad (ab_1, ab_2)^t \sim \mathbf{2} \;, \quad (a'b_1, -a'b_2)^t \sim \mathbf{2} \;,$$

$$\mathbf{1}^{(\prime)} \times \mathbf{3} : \qquad (ac_1, ac_2, ac_3)^t \sim \mathbf{3} \;, \quad (a'c_1, a'c_2, a'c_3)^t \sim \mathbf{3}' \;,$$

$$\mathbf{1}^{(\prime)} \times \mathbf{3}' : \qquad (ac'_1, ac'_2, ac'_3)^t \sim \mathbf{3}' \;, \quad (a'c'_1, a'c'_2, a'c'_3)^t \sim \mathbf{3} \;.$$

For a doublet coupled to a doublet

$$2 \times 2$$
: $b_1 \tilde{b}_2 + b_2 \tilde{b}_1 \sim 1$, $b_1 \tilde{b}_2 - b_2 \tilde{b}_1 \sim 1'$, $(b_2 \tilde{b}_2, b_1 \tilde{b}_1)^t \sim 2$.

For a doublet multiplied with a triplet

$$\mathbf{2} \times \mathbf{3}: \quad (b_1c_2 + b_2c_3, b_1c_3 + b_2c_1, b_1c_1 + b_2c_2)^t \sim \mathbf{3} \;, \quad (b_1c_2 - b_2c_3, b_1c_3 - b_2c_1, b_1c_1 - b_2c_2)^t \sim \mathbf{3}' \;,$$
 and

$$\mathbf{2} \times \mathbf{3}' : \quad (b_1 c_2' - b_2 c_3', b_1 c_3' - b_2 c_1', b_1 c_1' - b_2 c_2')^t \sim \mathbf{3} \;, \quad (b_1 c_2' + b_2 c_3', b_1 c_3' + b_2 c_1', b_1 c_1' + b_2 c_2')^t \sim \mathbf{3}' \;.$$

For the product 3×3

$$c_1\tilde{c}_1 + c_2\tilde{c}_3 + c_3\tilde{c}_2 \sim \mathbf{1} , \quad (c_1\tilde{c}_3 + c_2\tilde{c}_2 + c_3\tilde{c}_1, c_1\tilde{c}_2 + c_2\tilde{c}_1 + c_3\tilde{c}_3)^t \sim \mathbf{2} ,$$

$$(c_2\tilde{c}_3 - c_3\tilde{c}_2, c_1\tilde{c}_2 - c_2\tilde{c}_1, c_3\tilde{c}_1 - c_1\tilde{c}_3)^t \sim \mathbf{3} ,$$

$$(2c_1\tilde{c}_1 - c_2\tilde{c}_3 - c_3\tilde{c}_2, 2c_3\tilde{c}_3 - c_1\tilde{c}_2 - c_2\tilde{c}_1, 2c_2\tilde{c}_2 - c_1\tilde{c}_3 - c_3\tilde{c}_1)^t \sim \mathbf{3}' ,$$

as well as for the product $3' \times 3'$

$$\begin{aligned} &c_1'\tilde{c}_1' + c_2'\tilde{c}_3' + c_3'\tilde{c}_2' \sim \mathbf{1} \;, \;\; (c_1'\tilde{c}_3' + c_2'\tilde{c}_2' + c_3'\tilde{c}_1', c_1'\tilde{c}_2' + c_2'\tilde{c}_1' + c_3'\tilde{c}_3')^t \sim \mathbf{2} \;, \\ &(c_2'\tilde{c}_3' - c_3'\tilde{c}_2', c_1'\tilde{c}_2' - c_2'\tilde{c}_1', c_3'\tilde{c}_1' - c_1'\tilde{c}_3')^t \sim \mathbf{3} \;, \\ &(2c_1'\tilde{c}_1' - c_2'\tilde{c}_3' - c_3'\tilde{c}_2', 2c_3'\tilde{c}_3' - c_1'\tilde{c}_2' - c_2'\tilde{c}_1', 2c_2'\tilde{c}_2' - c_1'\tilde{c}_3' - c_3'\tilde{c}_1')^t \sim \mathbf{3}' \;, \end{aligned}$$

and finally for the product $3 \times 3'$

$$c_1c'_1 + c_2c'_3 + c_3c'_2 \sim \mathbf{1}' , \quad (c_1c'_3 + c_2c'_2 + c_3c'_1, -(c_1c'_2 + c_2c'_1 + c_3c'_3))^t \sim \mathbf{2} ,$$

$$(2c_1c'_1 - c_2c'_3 - c_3c'_2, 2c_3c'_3 - c_1c'_2 - c_2c'_1, 2c_2c'_2 - c_1c'_3 - c_3c'_1)^t \sim \mathbf{3} ,$$

$$(c_2c'_3 - c_3c'_2, c_1c'_2 - c_2c'_1, c_3c'_1 - c_1c'_3)^t \sim \mathbf{3}' .$$

These results are in accordance with [16].

Note that due to the choice of T being complex for the real representations $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$ for fields which transform as $(\phi_1, \phi_2)^t \sim \mathbf{2}$, $(\psi_1, \psi_2, \psi_3)^t \sim \mathbf{3}$ and $(\psi_1', \psi_2', \psi_3')^t \sim \mathbf{3}'$ their conjugates, $(\phi_1^\star, \phi_2^\star)^t$, $(\psi_1^\star, \psi_2^\star, \psi_3^\star)^t$ and $((\psi_1')^\star, (\psi_2')^\star, (\psi_3')^\star)^t$ are in $\mathbf{2}^\star$, $\mathbf{3}^\star$ and $(\mathbf{3}')^\star$, respectively, and only $(\phi_2^\star, \phi_1^\star)^t \sim \mathbf{2}$, $(\psi_1^\star, \psi_3^\star, \psi_2^\star)^t \sim \mathbf{3}$ and $((\psi_1')^\star, (\psi_3')^\star, (\psi_2')^\star)^t \sim \mathbf{3}'$ holds. Eventually, we display the embedding of S_4 into the continuous groups SO(3) and SU(3) as well as its breaking to the discrete groups A_4 and A_3 [41]. The smallest representations of SO(3) and SU(3) are decomposed into S_4 representations, respectively

SO(3)	\rightarrow	$\underline{S_4}$	SU(3)	\rightarrow	$\underline{S_4}$
1	\rightarrow	1	1	\rightarrow	1
3	\rightarrow	3	3	\rightarrow	3
5	\rightarrow	2 + 3 '	6	\rightarrow	1 + 2 + 3'
7	\rightarrow	1 '+ 3 + 3 '	8	\rightarrow	2 + 3 + 3 '
9	\rightarrow	1 + 2 + 3 + 3'	10	\rightarrow	1' + 3 + 3 + 3'

The decomposition of the irreducible representations of S_4 into those of the groups A_4 and S_3 leads to

Note that due to the choice of S, T and U the decomposition of S_4 representations into those of A_4 and S_3 can be nicely read off from the generators.

Particle	Σ	Σ^c	Δ	Δ^c	Υ	Υ^c	Ω	Ω^c	Θ	Θ^c
SU(5)	10	10	5	$\overline{5}$	5	$\overline{f 5}$	$\overline{5}$	5	$\overline{5}$	5
S_4	1	1	1	1	2	2	3	3	3	3
U(1)	-x-z	x+z	x + 2z	-x-2z	x	-x	y+2z	-y-2z	y+z	-y-z
U(1)	-6	6	7	-7	5	-5	6	-6	5	-5

Table 5: Heavy fields necessary to generate the diagrams given in figure 1. We list their U(1) charges in terms of the parameters (x, y, z) as well as for the specific case #13 where (x, y, z) = (5, 4, 1), as chosen in section 4.3. All fields carry a $U(1)_R$ charge +1.

B Messenger sector

As already discussed in section 2, in order to generate the operators $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$ and $(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3H_{\overline{5}}/M^3$ we have to require a specific choice of mediators to exist in a high energy completion of our effective theory. This is necessary in order to correctly achieve the GJ relations among the down quark and charged lepton masses as well as in order to ensure the validity of the GST relation. To this end, we add five pairs of heavy fields $\{\Sigma, \Sigma^c\}$, $\{\Delta, \Delta^c\}$, $\{\Upsilon, \Upsilon^c\}$, $\{\Omega, \Omega^c\}$ and $\{\Theta, \Theta^c\}$ which are vector-like under $SU(5) \times U(1)$. Apart from the pair $\{\Sigma, \Sigma^c\}$ all fields transform as 5-plets under SU(5). Similarly to the supermultiplets containing the SM particles, they carry a charge +1 under the $U(1)_R$ symmetry. Their transformation properties under the family symmetry $S_4 \times U(1)$ as well as under the SU(5) gauge group can be found in table 5. The U(1) charges are given in terms of the general parameters (x,y,z) as well as for the specific case #13 where (x,y,z)=(5,4,1), as chosen in section 4.3. The relevant terms in the superpotential which give rise to the first diagram of figure 1, and thus to $(F\widetilde{\Phi}_3^d)_1(T\Phi_2^d)_1H_{\overline{45}}/M^2$, are

$$w_{\text{heavy}} \supset \alpha_1 T \Phi_2^d \Sigma + \alpha_2 H_{\overline{45}} \Delta^c \Sigma^c + \alpha_3 \Delta F \widetilde{\Phi}_3^d$$
 (B.1)
 $+ M_{\Sigma} \Sigma^c \Sigma + M_{\Delta} \Delta^c \Delta$.

The second diagram of figure 1, corresponding to the operator $(F\Phi_2^d\Phi_2^d)_3(T\widetilde{\Phi}_3^d)_3H_{\overline{5}}/M^3$, is generated from the terms

$$w_{\text{heavy}} \supset \beta_1 T \Upsilon^c H_{\overline{5}} + \beta_2 \Upsilon \Omega \widetilde{\Phi}_3^d + \beta_3 \Omega^c \Theta \Phi_2^d + \beta_4 F \Theta^c \Phi_2^d$$

$$+ M_{\Upsilon} \Upsilon^c \Upsilon + M_{\Omega} \Omega^c \Omega + M_{\Theta} \Theta^c \Theta .$$
(B.2)

Note that we omit SU(5) indices throughout this calculation. According to [42] we can integrate out the heavy degrees of freedom $\{\Sigma, \Sigma^c\}$, $\{\Delta, \Delta^c\}$, $\{\Upsilon, \Upsilon^c\}$, $\{\Omega, \Omega^c\}$ and $\{\Theta, \Theta^c\}$ by computing the derivatives of w_{heavy} with respect to these fields, setting them to zero

and plugging the result for the heavy fields back into the superpotential w_{heavy} as well as the Kähler potential. The Kähler potential for the heavy fields is canonical in the lowest order in the expansion of flavon fields. We find for the derivatives of w_{heavy}

$$\frac{\partial w_{\text{heavy}}}{\partial \Sigma} = M_{\Sigma} \Sigma^{c} + \alpha_{1} (T_{1} \Phi_{2,2}^{d} + T_{2} \Phi_{2,1}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Sigma^{c}} = M_{\Sigma} \Sigma + \alpha_{2} H_{45} \Delta^{c},$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Delta} = M_{\Delta} \Delta^{c} + \alpha_{3} (F_{1} \tilde{\Phi}_{3,1}^{d} + F_{2} \tilde{\Phi}_{3,3}^{d} + F_{3} \tilde{\Phi}_{3,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Delta} = M_{\Delta} \Delta^{c} + \alpha_{3} (F_{1} \tilde{\Phi}_{3,1}^{d} + F_{2} \tilde{\Phi}_{3,3}^{d} + F_{3} \tilde{\Phi}_{3,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Delta^{c}} = M_{\Delta} \Delta + \alpha_{2} H_{45} \Sigma^{c},$$

$$\frac{\partial w_{\text{heavy}}}{\partial Y_{1}} = M_{\Upsilon} \Upsilon_{2}^{c} + \beta_{2} (\Omega_{1} \tilde{\Phi}_{3,2}^{d} + \Omega_{2} \tilde{\Phi}_{3,1}^{d} + \Omega_{3} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{1}^{c}} = M_{\Upsilon} \Upsilon_{2}^{c} + \beta_{2} (\Omega_{1} \tilde{\Phi}_{3,3}^{d} + \Omega_{2} \tilde{\Phi}_{3,2}^{d} + \Omega_{3} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{2}^{c}} = M_{\Upsilon} \Upsilon_{1}^{c} + \beta_{2} (\Omega_{1} \tilde{\Phi}_{3,3}^{d} + \Omega_{2} \tilde{\Phi}_{3,2}^{d} + \Omega_{3} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{2}^{c}} = M_{\Upsilon} \Upsilon_{1}^{c} + \beta_{1} H_{5} T_{2},$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{1}^{c}} = M_{\Upsilon} \Upsilon_{1}^{c} + \beta_{1} H_{5} T_{1},$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{2}^{c}} = M_{\Upsilon} \Upsilon_{1}^{c} + \beta_{1} H_{5} T_{1},$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{1}^{c}} = M_{\Upsilon} \Omega_{1}^{c} + \beta_{2} (\Upsilon_{1} \tilde{\Phi}_{3,2}^{d} + \Upsilon_{2} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{2}^{c}} = M_{\Theta} \Theta_{1}^{c} + \beta_{3} (\Omega_{1}^{c} \Phi_{2,1}^{d} + \Omega_{2}^{c} \Phi_{2,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{1}^{c}} = M_{\Upsilon} \Upsilon_{2}^{c} + \beta_{1} H_{5} T_{1},$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{1}^{c}} = M_{\Upsilon} \Omega_{1}^{c} + \beta_{2} (\Upsilon_{1} \tilde{\Phi}_{3,2}^{d} + \Upsilon_{2} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{1}^{c}} = M_{\Theta} \Theta_{1}^{c} + \beta_{4} (F_{1} \Phi_{2,1}^{d} + F_{2} \Phi_{2,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Upsilon_{2}^{c}} = M_{\Upsilon} \Omega_{1}^{c} + \beta_{2} (\Upsilon_{1} \tilde{\Phi}_{3,2}^{d} + \Upsilon_{2} \tilde{\Phi}_{3,3}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{2}^{c}} = M_{\Theta} \Theta_{1}^{c} + \beta_{4} (F_{1} \Phi_{2,1}^{d} + F_{2} \Phi_{2,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{1}^{c}} = M_{\Theta} \Theta_{2}^{c} + \beta_{4} (F_{1} \Phi_{2,1}^{d} + F_{2} \Phi_{2,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{2}^{c}} = M_{\Theta} \Theta_{2}^{c} + \beta_{4} (F_{1} \Phi_{2,1}^{d} + F_{2} \Phi_{2,2}^{d}),$$

$$\frac{\partial w_{\text{heavy}}}{\partial \Theta_{2}^{c}} = M_{\Theta} \Theta_{2}^{c} + \beta_{4} (F_{1} \Phi_{2,1}^{d} + F_{2} \Phi_{2,2}^{d}),$$

$$\frac{\partial$$

Plugging the solution for the heavy fields back into w_{heavy} and using the vacuum structure of Φ_2^d and $\widetilde{\Phi}_3^d$, as shown in Eq. (2.10), we arrive at

$$\frac{\alpha_1 \alpha_2 \alpha_3}{M_{\Lambda} M_{\Sigma}} (F_2 T_2 - F_3 T_2) H_{\overline{45}} \varphi_2^d \widetilde{\varphi}_3^d , \qquad (B.4)$$

$$\frac{\beta_1 \beta_2 \beta_3 \beta_4}{M_{\Omega} M_{\Theta} M_{\Upsilon}} (-F_2 T_1 + F_1 T_2 + F_3 T_1 - F_3 T_2) H_{\overline{5}} \varphi_2^d \varphi_2^d \widetilde{\varphi}_3^d , \qquad (B.5)$$

which shows that terms of exactly the required form are generated and no further ones, compare to Eqs. (2.3,2.4,2.11,2.12).

There are additional terms¹⁰ which also arise at the renormalisable level involving the messengers

 $\gamma_1 \widetilde{\Phi}_2^u \Upsilon \Upsilon^c + \gamma_2 \widetilde{\Phi}_2^u \Omega \Omega^c + \gamma_3 \widetilde{\Phi}_2^u \Theta \Theta^c . \tag{B.6}$

These terms are expected to give small corrections to the LO mass terms of the messengers which are M_{Υ} , M_{Ω} and M_{Θ} . The terms involving $\widetilde{\Phi}_{2}^{u}$ should be small compared to

 $^{^{10}}$ One can easily check that this set of additional renormalisable operators is exhaustive even if we consider the specific U(1) charges of case #13, see table 5.

those, since $\langle \widetilde{\Phi}_2^u \rangle \approx \lambda^4 M$ with M being the generic messenger mass. These terms can also be taken into account when integrating out the heavy fields and give a subleading contribution to the fermion mass matrices which is suppressed by λ^4 compared to the LO one. The flavour structure deviates from the one of the LO term. However, all such corrections have a small effect on the mass spectrum of the fermions and their mixings. Plugging the solution for the heavy fields into their Kähler potential shows that the non-canonical terms generated for the supermultiplets containing SM fermions are small and thus do not considerably affect our assumption of a canonical Kähler potential for all fields.

C List of unwanted terms with up to three flavons

Here we present all operators with up to three flavons which are classified either as dangerous or as marginal. As done in section 3 we only consider the case of k=1. Apart from the operator we show in this table also the λ -suppression of the contribution(s) due to this operator as well as the entries of the mass matrices which are in conflict with the LO setup. Entries for which the operator is marginal in the above sense are marked with square brackets, whereas for all other ones the operator is dangerous. Note that we only give one of the two entries (ij) and (ji) in the case of symmetric or symmetrised terms, TTH_5 , T_3TH_5 , NN. The three operators denoted with a prime (43', 48', 54') differ from the LO terms of the down quark sector in Eq. (2.2) only by the exchange of $H_{\overline{5}}$ and $H_{\overline{45}}$. All other terms given for the down quark sector must be forbidden for both Higgs fields, $H_{\overline{5}}$ as well as $H_{\overline{45}}$. Note that in this calculation we assumed the vacuum alignment of the flavons as given in Eqs. (2.6,2.10,2.21), apart from the fact that we allow $\langle \widetilde{\Phi}_3^d \rangle$ to be aligned as $(0, \kappa, 1)^t$ with $|\kappa| = 1$, κ complex, instead of using $(0, -1, 1)^t$ as shown in Eq. (2.10). ¹¹ Albeit these alignments lead to the same LO results for fermion masses and mixings, the results for the classification of dangerous and marginal operators are slightly different: the operator #18 which is dangerous becomes irrelevant for the specific alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$ as does the marginal operator #32. Note that, for notational simplicity, the appropriate powers of the messenger scale M, necessary to give the correct mass dimension of the operators, are omitted in the following table. For the operator $FTH_{5,\overline{45}}\Phi_{3'}^{\nu}/M$ (i1) indicates that the contributions to the (11), (21) as well as (31) elements are classified as dangerous.

¹¹Note that in the discussion of the flavon superpotential in section 4 we present a setup of driving fields which only leads to the alignment $\langle \widetilde{\Phi}_3^d \rangle \propto (0, -1, 1)^t$.

	TTH	I_5	
#	Operator	Structure	$\mathcal{O}(\lambda)$
1	$TTH_5\Phi_2^d$	(11)	λ
2	$TTH_5(\Phi_2^d)^2$	(22)	λ^2
3	$TTH_5(\Phi_2^d)^3$	(12)	λ^3
4	$TTH_5\Phi_2^{\nu}$	(11)[(22)]	λ^4
5	$TTH_5\Phi_1^{ u}$	(12)	λ^4
6	$TTH_5(\Phi_3^d)^2$	(11)	λ^4
7	$TTH_{5}\Phi_{3}^{d}\widetilde{\Phi}_{3}^{d}$	(11), (12)	λ^5
8	$TTH_5\Phi_2^u\Phi_2^d$	(12)	λ^5
9	$TTH_{5}\widetilde{\Phi}_{2}^{u}\Phi_{2}^{d}$ $TTH_{5}\Phi_{2}^{d}\Phi_{2}^{\nu}$	(12)	λ^5
10	$TTH_5\Phi_2^d\Phi_2^ u$	(12)	λ^5
11	$TTH_5\Phi_2^a\Phi_1^{\nu}$	(11)	λ^5
12	$TTH_5\Phi_3^d\Phi_{3'}^{ u}$	(11)	λ^6
13	$TTH_5(\widetilde{\Phi}_3^d)^2$	(11)[(12)]	λ^6
14	$TTH_5\Phi_3^d\widetilde{\Phi}_3^d\Phi_2^d$	(11)	λ^6
15	$TTH_5(\Phi_2^d)^2\Phi_2^{\nu}$	(11)[(12)]	λ^6
16	$TTH_5\Phi^u_2(\Phi^d_2)^2$	(11)	λ^6
17	$TTH_5\widetilde{\Phi}_2^u(\Phi_2^d)^2$	(11)	λ^6
18	$TTH_{5}\widetilde{\Phi}_{3}^{d}\Phi_{3'}^{ u}$	(11)	λ^7
19	$TTH_5\Phi^d_3\Phi^d_2\Phi^{\nu}_{3'}$	(11)	λ^7
20	$TTH_5(\widetilde{\Phi}_3^d)^2\Phi_2^d$	(11)	λ^7
21	$TTH_{5}\widetilde{\Phi}_{2}^{u}$	[(22)]	λ^4
22	$TTH_5(\Phi_2^u)^2$	[(11)]	λ^8
23	$TTH_5(\widetilde{\Phi}_2^u)^2$	[(11)]	λ^8
24	$TTH_5\Phi_2^u\Phi_2^{\nu}$	[(11)]	λ^8
25	$TTH_{5}\widetilde{\Phi}_{2}^{u}\Phi_{2}^{\nu}$	[(11)]	λ^8
26	$TTH_5(\Phi^{ u}_{3'})^2$	[(11)]	λ^8
27	$TTH_5(\Phi_2^{\nu})^2$	[(11)]	λ^8
28	$TTH_5\Phi_2^{\nu}\Phi_1^{\nu}$	[(11)]	λ^8
29	$TTH_{5}(\Phi^{d}_{3})^{2}\Phi^{\nu}_{3'}$	[(11)]	λ^8
30	$TTH_5(\Phi_3^d)^2\Phi_1^{\nu}$	[(11)]	λ^8
31	$TTH_5\Phi_3^d(\widetilde{\Phi}_3^d)^2$	[(11)]	λ^8
32	$TTH_5\widetilde{\Phi}_3^d\Phi_2^d\Phi_{3'}^{\nu}$	[(11)]	λ^8

	T_3TH_5									
#	Operator	Structure	$\mathcal{O}(\lambda)$							
33	$T_3TH_5\Phi_2^d$	(32)	λ							
34	$T_3TH_5(\Phi_2^d)^2$	(31)	λ^2							
35	$T_3TH_5\Phi_2^u$	[(31)]	λ^4							
36	$T_3TH_5\widetilde{\Phi}_2^u$	[(31)]	λ^4							
37	$T_3TH_5\Phi_2^{ u}$	[(31)]	λ^4							

	F_{-}^{\prime}	$\Gamma H_{\overline{5},\overline{45}}$	
#	Operator	Structure	$\mathcal{O}(\lambda)$
38	$FTH_{\overline{5},\overline{45}}\Phi_3^d$	(11), (22)	λ^2
39	$FTH_{\overline{5},\overline{45}}\widetilde{\Phi}_3^d$	(11), (12), (22), (31)	λ^3
40	$FTH_{\overline{5},\overline{45}}\Phi_3^d\Phi_2^d$	(21), (32)	λ^3
41	$FTH_{\overline{5},\overline{45}}\Phi^{ u}_{3'}$	(i1), (12)[(22), (32)]	λ^4
42	$FTH_{\overline{5},\overline{45}}(\Phi_3^d)^2$	(12), (31)	λ^4
43'	$FTH_{\overline{5}}\widetilde{\Phi}_3^d\Phi_2^d$	(11), (21)[(22), (32)]	λ^4
44	$FTH_{\overline{5},\overline{45}}\Phi_3^d(\Phi_2^d)^2$	(12), (31)	λ^4
45	$FTH_{\overline{5},\overline{45}}\Phi_2^d\Phi_{3'}^{\nu}$	(11)[(12),(21),(31)]	λ^5
46	$FTH_{\overline{5},\overline{45}}(\Phi_3^d)^2\Phi_2^d$	(11)	λ^5
47	$FTH_{\overline{5},\overline{45}}\Phi_3^d\widetilde{\Phi}_3^d$	[(12), (21), (31)]	λ^5
48'	$FTH_{\overline{45}}\widetilde{\Phi}_3^d(\Phi_2^d)^2$	[(12), (21), (31)]	λ^5
49	$FTH_{\overline{5},\overline{45}}(\widetilde{\Phi}_3^d)^2$	[(11)]	λ^6
50	$FTH_{\overline{5},\overline{45}}\Phi_3^d\Phi_{3'}^{\nu}$	[(11)]	λ^6
51	$FTH_{\overline{5},\overline{45}}\Phi_3^d\Phi_1^{ u}$	[(11)]	λ^6
52	$FTH_{\overline{5},\overline{45}}(\Phi_2^d)^2\Phi_{3'}^{\nu}$	[(11)]	λ^6
53	$FTH_{\overline{5},\overline{45}}\Phi_3^d\widetilde{\Phi}_3^d\Phi_2^d$	[(11)]	λ^6

	$FT_3H_{\overline{5},\overline{5}}$	45	
#	Operator	Structure	$\mathcal{O}(\lambda)$
54'	$FT_3H_{\overline{45}}\Phi_3^d$	[(33)]	λ^2
55	$FT_3H_{\overline{5},\overline{45}}\widetilde{\Phi}_3^d$	(23)	λ^3
56	$FT_3H_{\overline{5},\overline{45}}\Phi_3^d\Phi_2^d$	(13)	λ^3
57	$FT_3H_{\overline{5},\overline{45}}\widetilde{\Phi}_3^d\Phi_2^d$	(13)	λ^4
58	$FT_3H_{\overline{5},\overline{45}}\Phi_3^d(\Phi_2^d)^2$	[(23)]	λ^4
59	$FT_3H_{\overline{5},\overline{45}}\Phi_3^d\widetilde{\Phi}_3^d$	[(13)]	λ^5
60	$FT_3H_{\overline{5},\overline{45}}\Phi_2^d\Phi_{3'}^{\nu}$	[(13)]	λ^5
61	$FT_3H_{\overline{5},\overline{45}}\widetilde{\Phi}_3^d(\Phi_2^d)^2$	[(13)]	λ^5

		NN	
#	Operator	Structure	$\mathcal{O}(\lambda)$
62	$NN\Phi_2^d$	(12), (33)	λ
63	$NN(\Phi_2^d)^2$	(13), (22)	λ^2
64	$NN\Phi_3^d\Phi_2^d$	(11), (23)	λ^3
65	$NN(\Phi_2^d)^3$	(11), (23)	λ^3
66	$NN\Phi_2^u$	(13), (22)	λ^4
67	$NN\widetilde{\Phi}_2^u$	(13), (22)	λ^4
68	$NN(\Phi_3^d)^2$	(12), (33)	λ^4
69	$NN\widetilde{\Phi}_3^d\Phi_2^d$	(11), (13), (22), (23)	λ^4
70	$NN\Phi_3^d(\Phi_2^d)^2$	(12), (33)	λ^4

D Relations of flavon VEVs

D.1 Correlations among flavon VEVs

In this part of appendix D we detail the calculations which lead to the results given in section 4.3. If we want to couple additional driving fields, not already present in table 3, to the flavons, in order to correlate the flavon VEVs further such fields obviously have to couple to at least two operator structures with different flavon content. For this to work, the latter operators have to: (i) have identical U(1) charges, (ii) transform identically under S_4 and (iii) obviously have the same overall λ -suppression if we insert the assumed suppression of the occurring flavon scales as given in Eqs. (2.8,2.13,2.25). Furthermore, as already mentioned at length above, we avoid introducing new mass scales into the flavon superpotential at this stage (with the exception of case #10, see below and table 6).

In the case that the additional driving field furnishes a doublet or a triplet representation of S_4 , we have to ensure that the F-terms of all the components vanish for the LO vacuum structure of the flavons. The following example illustrates this issue: let us consider the U(1) charge assignment #10 and a driving field Z_3^{new} being a triplet 3 under S_4 . For Z_3^{new} having the U(1) charge +13 we find two flavon combinations $\widetilde{\Phi}_2^u \widetilde{\Phi}_3^d$ and $(\Phi_3^d)^3 \Phi_2^d/M^2$ which can couple to Z_3^{new} in order to form an invariant under $S_4 \times U(1)$. Furthermore, these combinations reveal the same λ -suppression (λ^7). However, inserting the vacuum alignment of Eqs. (2.6,2.10,2.21), we find that for $\langle \Phi_2^u \Phi_3^d \rangle$ the first as well as the third component of the triplet 3 are non-zero, whereas only the third component of 3 is non-zero for $\langle (\Phi_3^d)^3 \Phi_2^d/M^2 \rangle$. Thus, we cannot satisfy the requirement of vanishing F-terms for all components of Z_3^{new} unless we set some of the flavon VEVs to zero. 12 This discussion shows that the constraint of the vanishing of the F-terms of all components considerably reduces the possibilities of introducing additional driving fields, which give rise to correlations between the scales of flavon VEVs. Moreover, also for any new driving field which couples consistently to at least two terms with identical λ -suppression, we have to check that the same driving field does not couple to other terms which are less suppressed and thus would strongly perturb the desired correlation.

Restricting ourselves, for practical purposes, to operators whose order in λ is $\leq \lambda^9$ if the scales of the flavons according to Eqs. (2.8,2.13,2.25) are plugged in, the number of possible new driving fields which correlate the different scales φ_2^u , $\widetilde{\varphi}_2^u$, φ_3^d , $\widetilde{\varphi}_3^d$, φ_2^d and φ_1^{ν} narrows down to only a few. In particular, for our four U(1) symmetries we find the correlations in table 6. A few aspects are interesting to observe: first of all, notice that for solution #10 the additional driving field(s) has (have) to be neutral under the U(1) symmetry. For this reason a coupling with mass dimension two is allowed, if the field X_1^{new} is used. Furthermore, we find only a limited number of possible correlations among the VEVs. Especially for a given choice of U(1) charges x, y and z we find at

 $^{^{12}}$ Moreover, a detailed analysis shows that there exist two operators, $M_{Z_3^{\rm new}}Z_3^{\rm new}\widetilde{\Phi}_3^d$, with $M_{Z_3^{\rm new}}\sim \lambda^{x_M}M$ being an explicit mass scale, and $Z_3^{\rm new}\Phi_3^d\Phi_2^u$, arising respectively at λ^{3+x_M} and λ^6 , that also give non-vanishing contributions if the flavon vacuum structure at LO is employed. These contributions would perturb any possible correlation among the VEVs $\widetilde{\varphi}_2^u,\,\varphi_2^d,\varphi_3^d$ and $\widetilde{\varphi}_3^d$.

U(1) symmetry	#10	#13			#21	#25	
Driving field	$X_{1 \text{ or } 1'}^{\text{new}}$	$\widetilde{X}_{1'}^{\mathrm{new}}$	$X_{1 \text{ or } 1'}^{\text{new}}$	$X_{1 \text{ or } 1'}^{\text{new}}$	$X_{1 \text{ or } 1'}^{\text{new}} \text{ or } Z_{3 \text{ or } 3'}^{\text{new}}$	$X_{1 \text{ or } 1'}^{\text{new}}$	$X_{1 \text{ or } 1'}^{\text{new}}$
U(1) charge	0	15	18	25	10	6	20
Order in λ	λ^8	λ^9	λ^8	λ^9	λ^8	λ^8	λ^9
Correlation	C _{#10}	C#13	C' _{#13}		$C_{\#21}$	C _{#25}	

Table 6: The driving fields and the resulting correlations for the four successful charge assignments. The correlations are as follows, $C_{\#10}: M^2(\varphi_3^d)^2 \varphi_1^{\nu} \sim (\varphi_2^d)^4 \varphi_2^u \big[+ M_{X_1^{\rm new}}^2 M^3 \big];$ $C_{\#13}: M^2 \widetilde{\varphi}_2^u \widetilde{\varphi}_3^d \sim \varphi_2^d (\varphi_3^d)^3;$ $C_{\#13}: M \varphi_2^u \sim \varphi_2^d \widetilde{\varphi}_3^d;$ $C_{\#21}: M^2 \varphi_2^u \varphi_1^{\nu} \sim (\varphi_2^d)^2 (\widetilde{\varphi}_3^d)^2$ and $C_{\#25}: M \varphi_2^u \sim \varphi_2^d \widetilde{\varphi}_3^d.$ Note that for case #10 additionally an explicit mass term for the field $X_1^{\rm new}$ is allowed as it is a (trivial) singlet under S_4 and uncharged with respect to the U(1) symmetry. The size of $M_{X_1^{\rm new}}$ has to be chosen as $\lambda^4 M$.

most two distinct relations among the VEVs. In the case in which a certain relation can be reproduced through several different driving fields, e.g. in the case #21 in which $M^2\varphi_2^u\varphi_1^\nu\sim (\varphi_2^d)^2(\widetilde{\varphi}_3^d)^2$ can be achieved through four different driving fields, $X_{1\,\text{or}\,1'}^{\text{new}}$ or $Z_{3\,\text{or}\,3'}^{\text{new}}$, obviously only one of these can be added to the model because otherwise we would have to require ad hoc relations among the parameters in the superpotential to reconcile the results. For a similar reason it is also not possible to introduce a driving field Y_2^{new} in the case of solution #21 instead of $X_{1\,\text{or}\,1'}^{\text{new}}$ or $Z_{3\,\text{or}\,3'}^{\text{new}}$. Thirdly, notice that only for the U(1) charge assignment #13 we find two distinct additional relations among the scales of the flavon VEVs. One might argue that we can also achieve two non-trivial relations if, in the case #10, we make use of both fields, X_1^{new} and $X_{1'}^{\text{new}}$. (This is possible since the consequential conditions for the flavon VEVs differ by the term related to the mass scale $M_{X_1^{\text{new}}}^2$.) However, as stated in section 4.1 we avoid the introduction of such terms into the flavon superpotential at this stage. Therefore we focus on scenario #13 in the phenomenological analysis presented in section 5.

From table 6 we see that scenario #13 can lead to the relation

$$M\varphi_2^u \sim \varphi_2^d \widetilde{\varphi}_3^d$$
, (D.1)

through a new driving field transforming as $\mathbf{1}$ or $\mathbf{1}'$ under S_4 carrying either charge +18 or charge +25 under the U(1). Since the relation arises at order λ^8 in the former case we will include - without loss of generality - the S_4 singlet driving field X_1^{new} with charge +18 in our model. Additionally, we add the field $\widetilde{X}_{1'}^{\text{new}}$ with U(1) charge +15 giving rise to the correlation

$$M^2 \widetilde{\varphi}_2^u \widetilde{\varphi}_3^d \sim \varphi_2^d (\varphi_3^d)^3$$
 (D.2)

The additional terms in the superpotential read

$$\frac{1}{M} X_{1}^{\text{new}} \Phi_{2}^{u} (\Phi_{3}^{d})^{2} + \frac{1}{M^{2}} X_{1}^{\text{new}} \Phi_{2}^{d} \widetilde{\Phi}_{3}^{d} (\Phi_{3}^{d})^{2}$$

$$= \frac{1}{M} X_{1}^{\text{new}} \left[\Phi_{2,1}^{u} ((\Phi_{3,3}^{d})^{2} + 2\Phi_{3,1}^{d} \Phi_{3,2}^{d}) + \Phi_{2,2}^{u} ((\Phi_{3,2}^{d})^{2} + 2\Phi_{3,1}^{d} \Phi_{3,3}^{d}) \right]$$

$$+ \frac{1}{M^{2}} X_{1}^{\text{new}} \left[(\Phi_{2,1}^{d} \widetilde{\Phi}_{3,2}^{d} - \Phi_{2,2}^{d} \widetilde{\Phi}_{3,3}^{d}) ((\Phi_{3,1}^{d})^{2} - \Phi_{3,2}^{d} \Phi_{3,3}^{d})$$

$$+ (\Phi_{2,1}^{d} \widetilde{\Phi}_{3,3}^{d} - \Phi_{2,2}^{d} \widetilde{\Phi}_{3,1}^{d}) ((\Phi_{3,2}^{d})^{2} - \Phi_{3,1}^{d} \Phi_{3,3}^{d}) + (\Phi_{2,1}^{d} \widetilde{\Phi}_{3,1}^{d} - \Phi_{2,2}^{d} \widetilde{\Phi}_{3,2}^{d}) ((\Phi_{3,2}^{d})^{2} - \Phi_{3,1}^{d} \Phi_{3,3}^{d}) \right],$$

and

$$\begin{split} &\frac{1}{M}\,\widetilde{X}_{1'}^{\text{new}}\widetilde{\Phi}_{2}^{u}\Phi_{3}^{d}\widetilde{\Phi}_{3}^{d} + \frac{1}{M^{3}}\,\widetilde{X}_{1'}^{\text{new}}\Phi_{2}^{d}(\Phi_{3}^{d})^{4} \\ &= \frac{1}{M}\,\widetilde{X}_{1'}^{\text{new}}\,\left[\widetilde{\Phi}_{2,1}^{u}(\Phi_{3,1}^{d}\widetilde{\Phi}_{3,2}^{d} + \Phi_{3,2}^{d}\widetilde{\Phi}_{3,1}^{d} + \Phi_{3,3}^{d}\widetilde{\Phi}_{3,3}^{d}) - \widetilde{\Phi}_{2,2}^{u}(\Phi_{3,1}^{d}\widetilde{\Phi}_{3,3}^{d} + \Phi_{3,2}^{d}\widetilde{\Phi}_{3,2}^{d} + \Phi_{3,3}^{d}\widetilde{\Phi}_{3,1}^{d})\right] \\ &+ \frac{a_{1}^{\text{new}}}{M^{3}}\,\widetilde{X}_{1'}^{\text{new}}((\Phi_{3,1}^{d})^{2} + 2\Phi_{3,2}^{d}\Phi_{3,3}^{d})\left[\Phi_{2,1}^{d}((\Phi_{3,3}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,2}^{d}) - \Phi_{2,2}^{d}((\Phi_{3,2}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,3}^{d})\right] \\ &+ \frac{a_{2}^{\text{new}}}{M^{3}}\,\widetilde{X}_{1'}^{\text{new}}\left[\Phi_{2,1}^{d}((\Phi_{3,2}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,3}^{d})^{2} - \Phi_{2,2}^{d}((\Phi_{3,3}^{d})^{2} + 2\Phi_{3,1}^{d}\Phi_{3,2}^{d})^{2}\right]. \end{split}$$

Finally, one can check that the VEVs of the driving fields X_1^{new} and $\widetilde{X}_{1'}^{\text{new}}$, which are determined by the F-term equations of the flavons, vanish if the LO alignments of Eqs. (2.6,2.10,2.21) are applied.

D.2 Mass scales in the flavon superpotential

Here we introduce further driving fields which allow for additional mass scales in the flavon superpotential. Thus the trivial vacuum in which all flavon VEVs vanish can be destabilised. In addition, the number of free parameters among the undetermined VEVs, $\widetilde{\varphi}_2^u$, φ_3^d , φ_2^d , and φ_1^{ν} , can be reduced to a minimum of only one parameter. We consider only the specific case #13 which we have singled out in section 4.3 and in the first part of this appendix.

Invoking a driving field V_0 which is neutral under the family symmetry $S_4 \times U(1)$ allows for the following couplings up to order λ^8 (if we already take into account the phenomenologically determined sizes of the different flavon VEVs)

$$V_0 M_{V_0}^2 + V_0 (\widetilde{\Phi}_2^u)^2 + V_0 (\Phi_3^d)^2 \Phi_1^{\nu} / M + V_0 (\Phi_3^d)^2 \Phi_2^{\nu} / M + V_0 (\Phi_3^d)^2 \Phi_{3'}^{\nu} / M . \tag{D.1}$$

Studying the equation derived from the F-term of V_0 and using the LO form of the flavon VEVs, we find the relation

$$M_{V_0}^2 + (\varphi_3^d)^2 \varphi_2^{\nu} / M + (\varphi_3^d)^2 \varphi_{3'}^{\nu} / M = 0.$$
 (D.2)

As the VEVs φ_2^{ν} and $\varphi_{3'}^{\nu}$ are already related to φ_1^{ν} through the *F*-terms of the driving fields $Z_{3'}^{\nu}$ and Y_2^{ν} , see Eq. (4.9), we find that φ_3^d can be expressed through M_{V_0} and

 φ_1^{ν} . In order to achieve the correct order of magnitude of the VEV φ_3^d we demand that $M_{V_0} \sim \lambda^4 M$. More importantly, in order to fulfil Eq. (D.2) the flavon VEVs φ_1^{ν} and φ_3^d must be non-zero. This excludes the trivial solution with only vanishing VEVs which cannot be avoided if only the driving fields listed in the main text are present. Therefore V_0 ensures that the family symmetry actually gets broken.

In a similar way we can introduce a field $V_2 \sim (2, -8)$ which allows, up to the order λ^8 , for the following couplings

$$M_{V_2}V_2\Phi_2^{\nu} + V_2\widetilde{\Phi}_2^{u}\Phi_1^{\nu} + V_2\widetilde{\Phi}_2^{u}\Phi_2^{\nu} + V_2(\Phi_2^d)^8/M^6$$
 (D.3)

From the F-terms of the two components of V_2 , $V_{2,1}$ and $V_{2,2}$, we find the relations

$$M_{V_2}\varphi_2^{\nu} + \widetilde{\varphi}_2^u \varphi_1^{\nu} + (\varphi_2^d)^8 / M^6 = 0$$
, (D.4)

$$M_{V_2}\varphi_2^{\nu} + \widetilde{\varphi}_2^{\nu}\varphi_2^{\nu} = 0 , \qquad (D.5)$$

if we apply the LO results for the flavon VEVs. As one can see, we can relate the VEV $\widetilde{\varphi}_2^u$ to the mass scale M_{V_2} . In order to end up with the correct order of magnitude for $\widetilde{\varphi}_2^u$, we have to set $M_{V_2} \sim \lambda^4 M$. Using $\varphi_2^{\nu} \sim \varphi_1^{\nu}$, Eq. (D.4) additionally leads to a determination of φ_2^d in terms of φ_1^{ν} and M_{V_2} and consistently leads to $\varphi_2^d \sim \lambda M$. Notice further that the inclusion of the driving field V_2 is essential for giving non-zero VEVs to all flavon fields. (Here we are still assuming that a solution with $\langle \Phi_{3'}^{\nu} \rangle \neq 0$ is chosen, as shown in Eqs.(4.7,4.8).)

In summary, by adding the two further driving fields V_0 and V_2 we can enforce the breaking of the family symmetry, eliminate three of the four undetermined parameters among the flavon VEVs and ensure that all these VEVs have to be non-vanishing. The explicit mass scales, M_{V_0} and M_{V_2} , as well as the free parameter φ_1^{ν} all have to be of the order of $\lambda^4 M$ in order to generate the sizes of the flavon VEVs as invoked in the discussion of fermion masses and mixings in section 2. Obviously, we have to choose these values by hand. In order to fully include these fields into the model presented in section 5, a careful study of the subleading corrections arising from higher-dimensional operators as well as a re-calculation of the shifts in the flavon VEVs would have to be performed.

We have also studied the effect of other possible driving fields V allowing for terms of the form $M_VV\Phi$ with Φ being a flavon. However, several of these (i) cannot be consistently introduced, (ii) lead to some parameter fine-tuning if considered in a setup together with V_0 or (iii) lead to redundant results only. Therefore we conclude that the presented choice of fields, V_0 and V_2 , is the most favourable one.

Obviously, such fields could also be considered for the choices of U(1) charges which we have discarded in section 4, see Eq. (4.19). We have checked that a consistent introduction of such fields is generally possible, however, it does not lead to a scenario with less parameters than the one presented in the paper.

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